On Existence of Optimal Linear Encoders over Non-field Rings for Data Compression with Application to Computing

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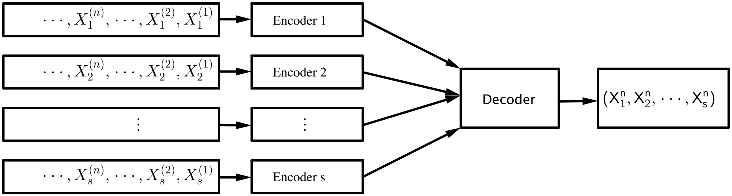
Linear Source Coding over Finite Fields / Rings (I)

Consider the Slepian–Wolf Source Network:



1. [Elias(1955), Csiszár(1982)] propose to use linear mappings (over finite fields) as encoders for Slepian–Wolf data compression;
2. Linear coding over finite fields (LCoF) is optimal, i.e. achieves the Slepian–Wolf region [Slepian and Wolf(1973)].

Linear Source Coding over Finite Fields / Rings (II)

How about linear coding over finite rings (LCoR)?

Definition 1

The tuple $[\mathfrak{R}, +, \cdot]$ is called a *ring* if the following criteria are met:

1. $[\mathfrak{R}, +]$ is an *Abelian group*;
2. There exists a *multiplicative identity* $1 \in \mathfrak{R}$, namely, $1 \cdot a = a \cdot 1 = a$, $\forall\, a \in \mathfrak{R}$;
3. $\forall\, a, b, c \in \mathfrak{R}$, $a \cdot b \in \mathfrak{R}$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
4. $\forall\, a, b, c \in \mathfrak{R}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

Examples: real (complex) numbers $\mathbb{R}$ ($\mathbb{C}$), integers, $\mathbb{Z}_q$ (q is any positive integer), polynomials, matrices and etc. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ (p is a prime), invertible matrices are fields.

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- † Will LCoR be optimal as LCoF for Slepian–Wolf coding?
- † What is the benefit?

Achievability Theorem (I)

Definition 2

A subset $\mathfrak{I}$ of a ring $[\mathfrak{R}, +, \cdot]$ is said to be a *left ideal* of $\mathfrak{R}$, denoted by $\mathfrak{I} \leq_l \mathfrak{R}$, if and only if

1. $[\mathfrak{I}, +]$ is a subgroup of $[\mathfrak{R}, +]$;
2. $\forall\, x \in \mathfrak{I}$ and $\forall\, a \in \mathfrak{R}$, $a \cdot x \in \mathfrak{I}$.

$\{0\}$ is a *trivial* left ideal, usually denoted by 0.

Examples: all even numbers of integers, $\{0, 2\}$ of $\mathbb{Z}_4$, the ring itself.

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Definition 3

Given a finite ring $\mathfrak{R}$ and one of its left ideal $\mathfrak{I}$, the *coset* $\mathfrak{R}/\mathfrak{I}$ is the set

$$\{r_1 + \mathfrak{I}, r_2 + \mathfrak{I}, \cdots, r_m + \mathfrak{I}\},$$

where $m = |\mathfrak{R}| \,/\, |\mathfrak{I}|$, $r_i \in \mathfrak{R}$ for all feasible i and $r_i + \mathfrak{I} \cap r_j + \mathfrak{I} = \emptyset \Leftrightarrow i \neq j$. $\mathfrak{R}/\mathfrak{I}$ forms a *left module* over $\mathfrak{R}$.

Achievability Theorem (II)

Assume that the sample space of X_i $(1 \leq i \leq s)$ is a finite set $\mathscr{X}_i$, and write

$$X_T = \prod_{i \in T} X_i, \; \mathfrak{R}_T = \prod_{i \in T} \mathfrak{R}_i$$

for $\emptyset \neq T \subseteq \{1, 2, \cdots, s\}$ and $\mathfrak{I}_T = \prod_{i \in T} \mathfrak{I}_i$ where $\mathfrak{I}_i \leq_l \mathfrak{R}_i$.

Let $\Phi = \{\Phi_1, \Phi_2, \cdots, \Phi_s\}$, where $\Phi_i : \mathscr{X}_i \rightarrow \mathfrak{R}_i$ is any injective mapping.

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Theorem 4 ([Huang and Skoglund(2013a)])

The region $\mathcal{R}_\Phi$ containting coding rate $(R_1, R_2, \cdots, R_s) \in \mathbb{R}^s$ that satisfies

$$\sum_{i \in T} \frac{R_i \log |\mathfrak{I}_i|}{\log |\mathfrak{R}_i|} > H(X_T | X_{T^c}) - H(Y_{\mathfrak{R}_T / \mathfrak{I}_T} | X_{T^c}), \tag{1}$$

$$\forall \, \emptyset \neq T \subseteq \{1, 2, \cdots, s\} \text{ and for all } 0 \neq \mathfrak{I}_i \leq_l \mathfrak{R}_i,$$

where $Y_{\mathfrak{R}_T / \mathfrak{I}_T} = \prod_{i \in T} \Phi_i(X_i) + \mathfrak{I}_T$ *(which has sample space $\mathfrak{R}_T / \mathfrak{I}_T$), is achievable with linear coding over* $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_s$.

Exist Optimal Linear Encoders over Non-field Rings (I)

Theorem 5 ([Huang and Skoglund(2013b)])

Let $\mathfrak{R}_1, \mathfrak{R}_2, \cdots, \mathfrak{R}_s$ be s finite rings with $|\mathfrak{R}_i| \geq |\mathscr{X}_i|$. If $\mathfrak{R}_i$ is isomorphic to either

1. *a field, i.e. $\mathfrak{R}_i$ contains no proper non-trivial left (right) ideal; or*
2. *a ring containing one and only one proper non-trivial left ideal $\mathfrak{I}_{0i}$ and $|\mathfrak{I}_{0i}| = \sqrt{|\mathfrak{R}_i|}$,*

for all feasible i, then the convex hull of $\bigcup_{\Phi} \mathcal{R}_{\Phi}$ coincides with the Slepian–Wolf region.

Examples: All finite fields, $\mathbb{Z}_{p^2}$ (p is a prime) and

$$\mathbb{M}_{L,p} = \left\{ \begin{bmatrix} a & 0 \\ b & a \end{bmatrix} a, b \in \mathbb{Z}_p \right\}.$$

Exist Optimal Linear Encoders over Non-field Rings (II)

Proof of Theorem 5 (for Single Source): There is nothing to prove if $\mathfrak{R}_1$ is a field. Assume that $\mathfrak{R}_1$ is a non-field ring. Then $\bigcup_{\Phi} \mathcal{R}_\Phi$ is the Slepian–Wolf region if and only if there exists $\tilde{\Phi}_1 : \mathscr{X}_1 \to \mathfrak{R}_1$ such that

$$\frac{\log |\mathfrak{R}_1|}{\log |\mathfrak{I}_{01}|}[H(X_1) - H(\tilde{\Phi}_1 + \mathfrak{I}_{01})] \leq H(X_1) \tag{2}$$

$$\Leftrightarrow H(X_1) \leq 2H(\tilde{\Phi}_1 + \mathfrak{I}_{01}) \qquad \left(\text{since } \sqrt{|\mathfrak{R}_1|} = |\mathfrak{I}_{01}|\right). \tag{3}$$

The existence of such a injection $\tilde{\Phi}_1$ is guaranteed by Lemma 6. ■

Exist Optimal Linear Encoders over Non-field Rings (III)

Lemma 6 ([Huang and Skoglund(2012)])

Let $\mathfrak{R}$ be a finite ring, X and Y be two correlated discrete random variables, and $\mathscr{X}$ be the sample space of X with $|\mathscr{X}| \leq |\mathfrak{R}|$. If $\mathfrak{R}$ contains one and only one proper non-trivial left ideal $\mathfrak{I}$ and $|\mathfrak{I}| = \sqrt{|\mathfrak{R}|}$, then there exists injection $\tilde{\Phi} : \mathscr{X} \to \mathfrak{R}$ such that

$$H(X|Y) \leq 2H(\tilde{\Phi}(X) + \mathfrak{I}|Y). \quad (4)$$

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Sketch of the Proof: Let $\tilde{\Phi} = \arg\max_{\Phi} H(\Phi(X) + \mathfrak{I}|Y)$. By the grouping rule for entropy, there exists $\overline{\Phi} : \mathscr{X} \to \mathfrak{R}$ such that

$$H(X|Y) - H(\tilde{\Phi}(X) + \mathfrak{I}|Y) = H(\overline{\Phi}(X) + \mathfrak{I}|Y).$$

Since

$$H(\tilde{\Phi}(X) + \mathfrak{I}|Y) \geq H(\overline{\Phi}(X) + \mathfrak{I}|Y)$$

by definition, the statement follows. ■

Other Rings

Example 7

Consider the single source scenario, where $X_1 \sim p$ and $\mathscr{X}_1 = \mathbb{Z}_6$, specified as follows.

X_1	0	1	2	3	4	5
$p(X_1)$	0.05	0.1	0.15	0.2	0.2	0.3

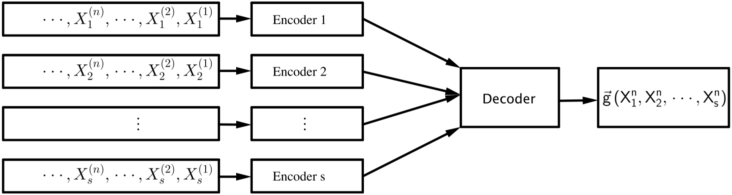
By Theorem 4,

$$\begin{aligned} \mathcal{R} &= \{R_1 \in \mathbb{R} | R_1 > \max\{2.40869, 2.34486, 2.24686\}\} \\ &= \{R_1 \in \mathbb{R} | R_1 > 2.40869 = H(X_1)\} \end{aligned}$$

is achievable with linear coding over ring $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$. Obviously, $\mathcal{R}$ is just the Slepian–Wolf region. Optimality is claimed.

Source Coding for Computing

Source Coding for Computing g (a discrete function):



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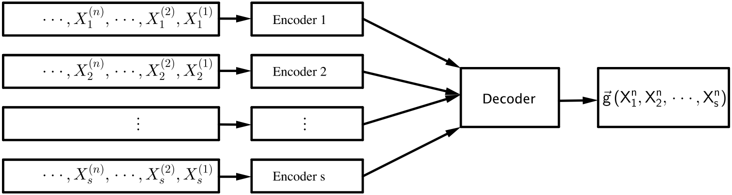

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One trick: Let $Z^{(n)} = g\left(X_1^{(n)}, X_2^{(n)}, \cdots, X_s^{(n)}\right)$ and ϕ be a linear encoder (over some field / ring) such that

$$\begin{aligned}\epsilon &> \Pr\{\psi(\phi(Z^n)) \neq Z^n\} \\ &\overset{(a)}{=} \Pr\{\psi(\vec{g}(\phi(X_1^n), \phi(X_2^n), \cdots, \phi(X_s^n))) \neq Z^n\},\end{aligned}$$

where (a) holds when g is a linear function over some field / ring.

LCoF is not optimal in the Sense of [Körner and Marton(1979)] (I)

Consider linear function over $\mathbb{Z}_4$

$$g(x, y, z) = x + 2y + 3z \text{ defined on the domain } \{0, 1\}^3 \subsetneq \mathbb{Z}_4^3.$$

g can also be presented as polynomial function

$$\hat{h}(x + 2y + 4z) \text{ defined on domain } \{0, 1\} \subsetneq \mathbb{Z}_5^3,$$

where

$$\hat{h}(x) = \sum_{a \in \mathbb{Z}_5} a \left[1 - (x - a)^4\right] - \left[1 - (x - 4)^4\right]$$

is not injective. Linear coding (LC) techniques (over non-field ring $\mathbb{Z}_4$ or field $\mathbb{Z}_5$) are used for encoding g.
However, the achievable region $\mathcal{R}_{\mathbb{Z}_4}$ achieved linear LC over $\mathbb{Z}_4$ always dominates the one $\mathcal{R}_{\mathbb{Z}_5}$ achieved by LC over $\mathbb{Z}_5$. In fact, $\mathcal{R}_{\mathbb{Z}_4}$ dominates the region achieved by LC over each and every finite field for encoding g.

LCoF is not optimal in the Sense of [Körner and Marton(1979)] (II)

Definition 8

The *characteristic* of a finite ring $\mathfrak{R}$ is defined to be the smallest positive integer m, such that $\sum_{j=1}^{m} 1 = 0$, where 0 and 1 are the zero and the multiplicative identity of $\mathfrak{R}$, respectively.

LCoF is not optimal in the Sense of [Körner and Marton(1979)] (II)

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The essential reason for the "domination" to happen is due to the fact that:

1. the characteristic of a finite field much be a prime (by the theory of splitting field);
2. the characteristic of a finite ring can be any integer ≥ 2.

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1. the characteristic of a finite field much be a prime (by the theory of splitting field);
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Basic on this fact, one can construct infinitely many functions, say g, such that LC over a finite field is always suboptimal (in the sense of [Körner and Marton(1979)]) for encoding g.

Non-field Ring vs Field

	field	non-field ring	properties
Slepian–Wolf coding (side information)	$\surd$	exist optimal encoders for all scenarios	inverse & typicality lemma
Slepian–Wolf coding (memory[1])	$\surd$	not yet proved, optimal shown by examples	inverse & typicality lemma
Implementation Complexity		$\surd$	polynomial long division algorithm
Alphabet sizes of encoders		$\surd$	prime subfield
Coding for Computing		$\surd$	characteristic & zero divisor
Coding for Computing (memory)		$\surd$	characteristic & zero divisor

[1] Please kindly refer to [Huang and Skoglund(2013c)] for details.

Thanks

♫ ♬

Thanks!

Bibliography I

- J. Körner and K. Marton, "How to encode the modulo-two sum of binary sources," *IEEE Transactions on Information Theory*, vol. 25, no. 2, pp. 219–221, Mar. 1979.
- P. Elias, "Coding for noisy channels," *IRE Conv. Rec.*, pp. 37–46, Mar. 1955.
- I. Csiszár, "Linear codes for sources and source networks: Error exponents, universal coding," *IEEE Transactions on Information Theory*, vol. 28, no. 4, pp. 585–592, Jul. 1982.
- D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Transactions on Information Theory*, vol. 19, no. 4, pp. 471–480, Jul. 1973.
- S. Huang and M. Skoglund, "On achievability of linear source coding over finite rings," in *IEEE International Symposium on Information Theory*, July 2013.

Bibliography II

- ——, "On existence of optimal linear encoders over non-field rings for data compression with application to computing," in *IEEE Information Theory Workshop*, September 2013.
- ——, *On Existence of Optimal Linear Encoders over Non-field Rings for Data Compression*, KTH Royal Institute of Technology, December 2012. [Online]. Available: http://www.ee.kth.se/~sheng11
- R. Ahlswede and T. S. Han, "On source coding with side information via a multiple-access channel and related problems in multi-user information theory," *IEEE Transactions on Information Theory*, vol. 29, no. 3, pp. 396–411, May 1983.
- S. Huang and M. Skoglund, *Coding for Computing Irreducible Markovian Functions of Sources with Memory*, KTH Royal Institute of Technology, May 2013. [Online]. Available: http://www.ee.kth.se/~sheng11