Solutions

These are solutions to some problems given as exercise. There might be other solutions, and these might not be complete.

Exercise 17: Show that for all $n \in \mathbb{N}_0$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad .$$

Proof: We will show the above using the *Well Ordering Principle.*
Assume the set of counterexamples

$$C := \left\{ n \in \mathbb{N}_0 \;\middle|\; \sum_{k=0}^{n} k^2 \neq \frac{n(n+1)(2n+1)}{6} \right\}$$

is nonempty. By the Well Ordering Principle it contains a minimal element $m \in C$. The element m is not zero, as

$$\sum_{k=0}^{0} k^2 = 0^2 = 0 = \frac{0 \cdot (0+1) \cdot (2 \cdot 0 + 1)}{6} \quad .$$

Therefore, $m-1$ is a natural number ($m-1 \in \mathbb{N}_0$), and it can not be in C, as it is smaller than m. Thus

$$\begin{aligned} m^2 + \sum_{k=0}^{n} k^2 &= m^2 + \frac{(m-1) \cdot m \cdot (2m-1)}{6} \\ &= \frac{6m^2 + m \cdot (2m^2 - 3m + 1)}{6} \\ &= \frac{m \cdot (6m + 2m^2 - 3m + 1)}{6} \\ &= \frac{m \cdot (2m^2 + 3m + 1)}{6} \\ &= \frac{m \cdot (m+1) \cdot (2m+1)}{6} \end{aligned}$$

Contradicting that $m \in C$. Hence, the set C is empty and

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers $n \in \mathbb{N}_0$.

Here is another solution:
Proof by induction. Our induction *hypothesis* is the predicate

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \text{ for the number } n.$$

Which is true for the *base case* $n = 0$, as

$$\sum_{k=0}^{0} k^2 = 0^2 = 0 = \frac{0 \cdot (0+1) \cdot (2 \cdot 0 + 1)}{6} \quad .$$

Assume that the hypothesis holds for n, then

$$\begin{aligned}
\sum_{k=0}^{n+1} k^2 &= (n+1)^2 + \sum_{k=0}^{n} k^2 \\
&\overset{\text{hyp.}}{=} (n+1)^2 + \frac{n \cdot (n+1) \cdot (2n+1)}{6} \\
&= \frac{6(n+1)^2 + (n+1) \cdot (2n^2 + n)}{6} \\
&= \frac{(n+1) \cdot (2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1) \cdot (2n^2 + 4n + 3n + 6)}{6} \\
&= \frac{(n+1) \cdot (n+2) \cdot (2n+3)}{6}
\end{aligned}$$

Which shows that the hypothesis holds for $n + 1$, This is the *inductive step.*
We *conclude* by induction that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all natural numbers $n \in \mathbb{N}_0$.