Math 314: Discrete Mathematics **Spring 2019**
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Exercises

The following is a short list of exercises and practice problems that have been raised during the lecture. It might be incomplete.

Exercise 1: Let a,b, $c \in \mathbb{Z}$. Show if $a|b$ and $b|c$ then $a|c$.

Exercise 2: Let $A \subseteq D$ be a subset of D. Find a simpler expression for $\overline{\overline{A}}$ and $\overline{\emptyset} \subseteq D$. How many elements does $\{\emptyset, \{\emptyset\}\}$ has?

Exercise 3: Show that $A \setminus B = A \cap \overline{B}$.

Task: Read section 1.9.

Exercise 4: Alice says Bob lies. Bob says Caro lies. Caro says Alice and Bob lie. Who is telling the truth and who is a liar?

Exercise 5: Verfy that $(P \implies Q) \wedge (Q \implies P)$ is equivalent to $P \iff Q$.

Exercise 6: Decide whether or not $\forall x \, \exists y : x - 3y = 0$ holds true when the variables range over the natural numbers $\mathbb{N}$, integers $\mathbb{Z}$ or real numbers $\mathbb{R}$.

Exercise 7: Complete the table below.

The binary relation R is ... if and	only if its inverse relation R^{-1} is ...
total	
a partial function	
injective	
surjective	
bijective	

Exercise 8: Which properties has a relation R from D to C if $R \circ R^{-1}$ is a partial function and $(R \circ R^{-1})(x) = x$ for all $x \in C$?

Exercise 9: How many elements has the set pow($\{1, 2, 3\}$)? List them all.

Exercise 10: Decide whether the total function $f : \mathbb{R} \to \mathbb{R}$ is a bijection, surjection, injection when

i) $x \mapsto x + 2$

ii) $x \mapsto 2x$

iii) $x \mapsto x^3$

iv) $x \mapsto \sin(x)$

v) $x \mapsto x \cdot \sin(x)$

vi) $x \mapsto \exp(x) = e^x$

Exercise 11: Find an example of a (total) injective function from a set D to itself that is not surjective.

Exercise 12: Alice thinking of a number between 1 and 1000. What is the least number of yes/no questions you could ask her and be guaranteed to discover what number it is?

Exercise 13: Let $X = \{x_1, x_2 \ldots, x_6\}$.

a) How many subsets of X contain x_1?

b) How many subsets contain x_2 and x_3 but not x_5?

Exercise 14: Let A, B, C be sets. Find a general formula for $|A \cup B \cup C|$ and prove it.

Exercise 15: In Poker a hand consists of 5 cards.

a) How many hands are there?

b) How many with four-of-a-kind?

c) How many full houses?

d) How many with two pairs (not counting hands with four-of-a-kind or full-houses)?

Exercise 16: Show that a n-set has $n!$ permutations.

Exercise 17: Show that for all $n \in \mathbb{N}_0$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad .$$

Task: Read section 5.1.3 and 5.1.4.

Exercise 18: Show by induction that for all $n \in \mathbb{N}_0$ and $r \in \mathbb{C}$, $r \neq 1$

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} \quad .$$

Exercise 19: Show that for all $n \in \mathbb{N}$

$$\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n} \quad .$$

Exercise 20: Give an example of a relation for each combination of the following:

being a partial function, *total*, *injective*, and *surjective*.

In total these are 16 relations. For which combinations can you provide examples with a finite domain that equals the codomain?

Exercise 21: Let $f\colon A \to B$ be a (total) function and A be a finite set. Fill the boxes with $\geq$, $\leq$ or $=$ to receive the strongest correct statements:

a) $|f(A)| \;\square\; |B|$

b) If f is surjective, then $|A| \;\square\; |B|$

c) If f is surjective, then $|f(A)| \;\square\; |B|$

d) If f is injective, then $|f(A)| \;\square\; |A|$

e) If f is a bijection, then $|A| \;\square\; |B|$

Exercise 22: Alice wants to prove by induction that a predicate P holds for a certain natural number. She has proven that for all natural numbers

$$P(n) \implies P(n+3) \quad .$$

Suppose Alice proves that $P(5)$ holds. Which of the following propositions can she infer?

a) $P(n)$ holds for all $n \geq 5$

b) $P(3n)$ holds for all $n \geq 5$

c) $P(n)$ holds for $n = 8, 11, 14, \ldots$

d) $P(n)$ holds for all $n < 5$

e) $\forall n : P(3n+5)$

f) $\forall n > 2 : P(3n-1)$

g) $P(0) \implies \forall n : P(3n+2)$

h) $P(0) \implies \forall n : P(3n)$

Which of the following could Alice prove in order to conclude that $P(n)$ holds for all $n \geq 5$?

i) $P(0)$	v) $P(5)$, $P(6)$ and $P(7)$
ii) $P(5)$	vi) $P(2)$, $P(4)$ and $P(5)$
iii) $P(5)$ and $P(6)$	vii) $P(2)$, $P(4)$ and $P(6)$
iv) $P(0)$, $P(1)$ and $P(2)$	viii) $P(3)$, $P(5)$ and $P(7)$

Exercise 23: Compute 7^{10} per hand in two ways, naively as $7 \cdot (7 \cdot (7 \cdot (\dots)))$ and with the fast exponentiation method $(7 \cdot (7^2)^2)^2$.

Exercise 24: Find a state machine that models the Towers of Hanoi problem.

Exercise 25: Implement the reccursice algorithm that solves the Towers of Hanoi problem.

Exercise 27: Solve the linear recurrence

$$b_1 = 1 \text{ and } b_n = 3b_{n-1} + 5 \text{ if } n \geq 2.$$

Hint: Use Exercise 18 with $r = 3$.

Exercise 28: Show that for all $n \geq 1$

$$f_{n-1} \cdot f_{n+1} - f_n^2 = (-1)^n$$

where f_k is the k-th Fibonacci number.

Exercise 29: Show that for all $n \in \mathbb{N}_0$

$$\sqrt{5} \cdot f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Hint: The number $\frac{1 \pm \sqrt{5}}{2}$ is a root of the polynomial $x^2 - x - 1$.

Exercise 30: Implement Merge Sort.

Exercise 31: Start with 102 coins on a table, 98 showing heads and 4 showing tails. There are two ways to change the coins:

1. flip over any ten coins, or
2. place $n+1$ additional coins, all showing tails, on the table, where n is the number of coins showing heads.

a) Model the above as a state machine.

b) Explain how to reach a state with exactly one tail showing.

c) Define the following derived variables.
coins on the table, # heads showing, # tails showing, and the parity of each of those three.

Which variables are strictly increasing, weakly increasing, strictly decreasing, weakly deacreasing or constant?

d) Prove that it is not possible to reach a state which is exactly one haead showing.

Exercise 32: Give an example of a stable matching of 3 boys and 3 girls where no one gets their first choice.

Exercise 33: Let $c \in A$ be a character and $s, t \in A^*$ be strings. Prove $\#_c(s \cdot t) = \#_c(s) + \#_c(t)$ using structural induction.

Exercise 34: Evaluate the Ackermann function $A : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ at $(3, 3)$.

Exercise 35: Evaluate $\texttt{eval}(\texttt{subst}([3 * x], [x * [x + [-1]]]), 2)$ and $\texttt{eval}([x * [x + [-1]]], \texttt{eval}([3 * x], 2))$.

Exercise 36: Prove the two missing constructor cases ($[e_1 * e_2]$ and $[-e_1]$) of the Theorem that says $\texttt{eval}(\texttt{subst}(f, e, n) = \texttt{eval}(f, \texttt{eval}(e, n))$.

Exercise 37: Define reccussivly the sets $\{2^a \cdot 3^b \cdot 5^c \in \mathbb{N} \mid a, b, c \in \mathbb{N}_0\}$ and $\{2^{a+b} \cdot 3^b \cdot 5^{a+c} \in \mathbb{N} \mid a, b, c \in \mathbb{N}_0\}$.

Exercise 38: Let F_1 be the set with $5 \in F_1$ and if $n \in F_1$, then $5 \cdot n \in F_1$. Let F_2 be the set with $5 \in F_2$ and if $n, m \in F_2$, then $n \cdot m \in F_2$.

a) Show that one of these definitions is technically ambiguous.

b) Briefly explain what advantage unambiguous recursive definitions have over ambiguous ones.

c) A way to prove that $F_1 = F_2$, is to show first that $F_1 \subseteq F_2$ and second that $F_2 \subseteq F_1$. One of these containments follows easily by structural induction. Which one? What would be the induction hypothesis? (You do not need to complete a proof.)

Exercise 39: Prove that if $a \equiv b \mod n$ and $c \equiv d \mod n$, then $a+c \equiv b+d \mod n$ and $a \cdot c \equiv b \cdot d \mod n$.

Exercise 40: Show that an integer $n = \sum_{k=0} a_k \cdot 10^k$ is a multiple of 3 if and only if 3 divides the cross sum $\sum_{k=0} a_k$.

Exercise 41: Show that an integer $n = \sum_{k=0} a_k \cdot 10^k$ is a multiple of 11 if and only if 11 divides the alternatig cross sum $\sum_{k=0}(-1)^k \cdot a_k$.

Exercise 42: Find the inverse of $[5]_{11}$ in $\mathbb{Z}_{11}$.

Exercise 43: Show that beeing isomorphic defines an equivalence relation on all graphs (on n nodes).

Exercise 44: What is the maximal number of edges that a bipartite graph on $L \cup R$ with $|L| = k$ and $|R| = n - k$ has?

Exercise 45: Argue that in a non-empty regular bipartite graph $|L|$ is equal to $|R|$.

Exercise 46: Let $n \in \mathbb{N}$. Show that $\chi(C_{2n}) = 2$, $\chi(C_{2n+1}) = 3$ and $\chi(K_n) = n$. Moreover, find the chromatic number for the empty graph and the line graph L_n on n nodes.

Exercise 47: Show that a tree with at least two nodes has a leave, i.e., a node of degree 1.

Exercise 48: Among connected graphs whose sum of node degrees is 20

- What is the lagest possible number of nodes of such a graph?
- What is the smallest possible number of nodes of such a graph?