Dressians and Tropical Grassmannians

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Tropicalization of Linear Spaces

Grassmannian $\mathrm{Gr}_{\mathbb{K}}(d,n)$ ▶ the vanishing locus of the Plücker ideal $I_{d,n} \otimes_{\mathbb{Z}} \mathbb{K}$ parametrizes all d-dimensional linear spaces in $\mathbb{K}^n$

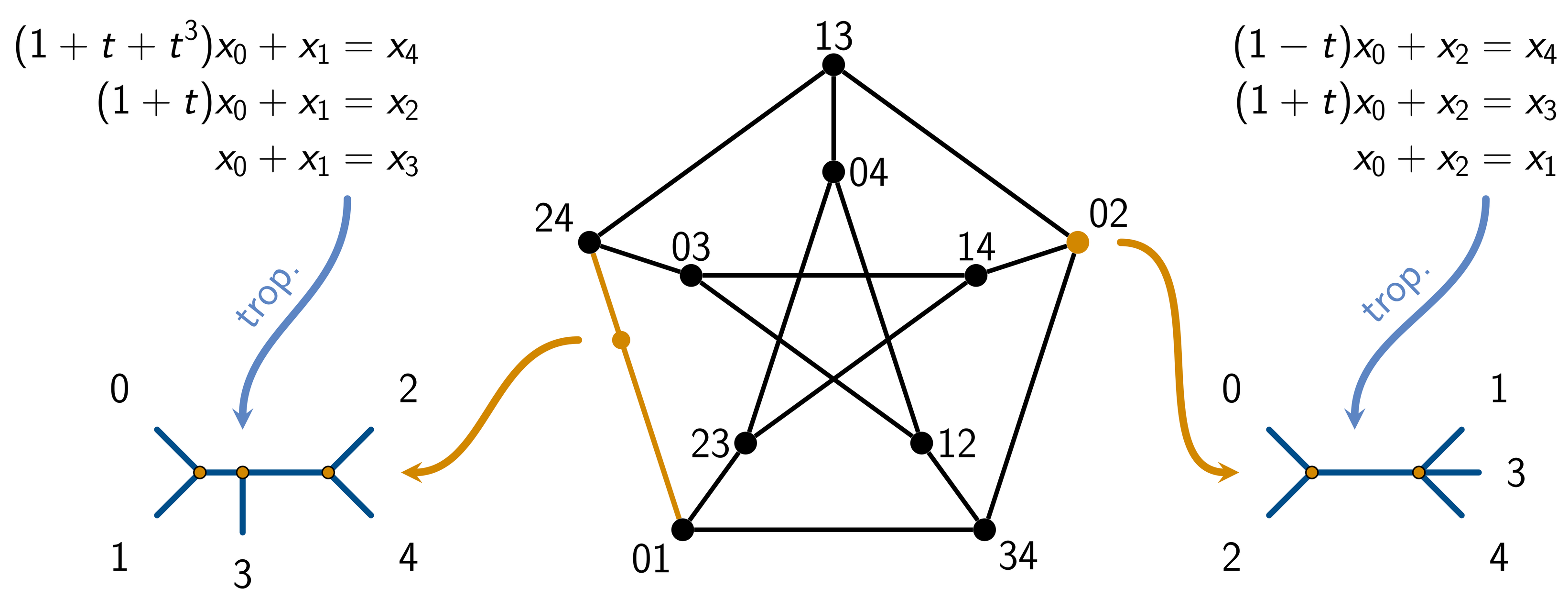
tropicalization ▶ subfan of the Gröbner fan of a homogeneous ideal I, containing exactly those cones which do not contain any monomial in their initial ideal $\mathrm{in}_\pi(I)$

tropical Grassmannian $\mathrm{TGr}_p(d,n)$ ▶ the tropicalization of $\mathrm{Gr}_{\mathbb{K}}(d,n)$ only depends on $p = \mathrm{char}\,\mathbb{K}$. This is a $(d(n-d)+1)$-dimensional pure polyhedral fan with a n-dimensional lineality space

[Speyer & Sturmfels 2004]

▶ There is a bijection between the points of the tropical Grassmannian $\mathrm{TGr}_p(d,n)$ and the collection of polyhedral complexes occurring as tropicalization of a d-plane in $\mathbb{K}^n$.
[Speyer & Sturmfels 2004]

The combinatorics of $\mathrm{TGr}_0(2,5)$ induced by formal Puiseux series $\mathbb{C}\{\{t\}\}$:



Matroidal Subdivisions

Dressian $\mathrm{Dr}(d,n)$ ▶ tropical prevariety generated by all 3-term Plücker relations

$$p_{Sab}p_{Scd} - p_{Sac}p_{Sbd} + p_{Sad}p_{Sbc} \overset{trop.}{\rightsquigarrow} \min\{\pi_{Sab}+\pi_{Scd}, \pi_{Sac}+\pi_{Sbd}, \pi_{Sad}+\pi_{Sbc}\}$$

This minimum is attained at least twice. The elements π of the Dressian are called tropical Plücker vectors.

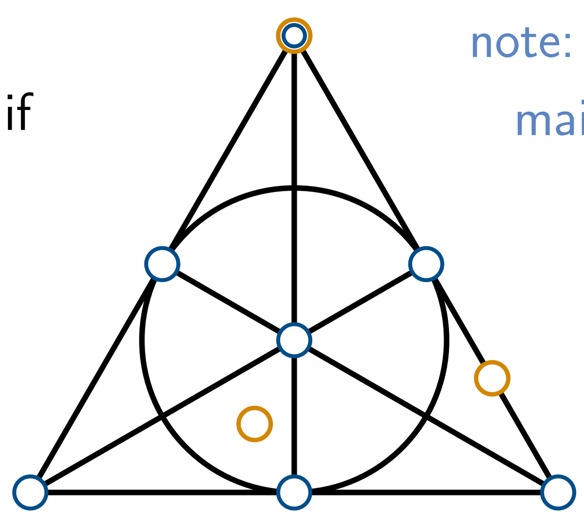
matroid ▶ abstraction of independence for a finite set $[n]$. Maximal independent subsets of $[n]$ are called bases. Any two bases B_1, B_2 satisfy the (strong) basis exchange property:

$$\forall a \in B_1 \setminus B_2 \; \exists b \in B_2 \setminus B_1 \; : \; b \cup B_1 \setminus a \text{ and } a \cup B_2 \setminus b \text{ are bases}$$

The elements of rank 0 in a matroid are called loops.

Fano matroid ▶ important example of a matroid, which is realizable if and only if $p = 2$. Therefore there is a tropical Plücker vector in $\mathrm{Dr}(3,8)$ that is not contained in $\mathrm{TGr}_0(3,8)$.

matroid polytope ▶ convex hull of the characteristic vectors of all bases of a matroid. The hypersimplex is the matroid polytope of the uniform matroid.



The rank 3 extensions of the Fano matroid.

▶ The Dressian $\mathrm{Dr}(d,n)$ is the subfan of the secondary fan of the hypersimplex $\Delta(d,n)$ containing exactly those vectors that subdivide the hypersimplex into matroid polytopes.
[Speyer 2005]

tropical linear space ▶ all vectors $u \in \mathbb{R}^n$ such that all sets S, for which $\pi_S + \sum_{j \in S} u_j$ is minimal, form the bases of a loop-free matroid.

We did our computations in `polymake` 2.14 and `Singular` 4.0.2.

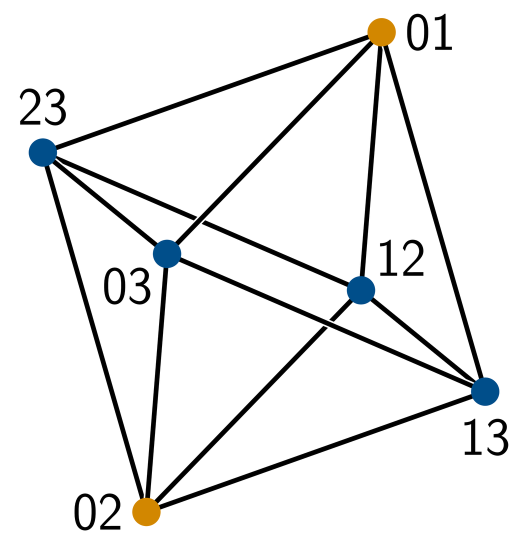
The Dimension of the Dressian

Theorem A (J.&S. 2015+)

For fixed d the dimension of $\mathrm{Dr}(d,n)$ is of order $\Theta(n^{d-1})$.

Sketch of the Proof:

- A stable set of size s of the Johnson graph, i.e. vertex-edge graph of the hypersimplex $\Delta(d,n)$, gives rise to a regular matroid subdivision of $\Delta(d,n)$ with s facets. [Herrmann, Jensen, Joswig & Speyer 2009]
- stable sets with $\geq \frac{1}{n}\binom{n}{d}$ nodes exist [Knuth 1974]
- number of facets in a matroid subdivision of $\Delta(d,n)$ is at most $\binom{n-2}{d-1}$ [Speyer 2005]



The Tropical Grassmannian $\mathrm{TGr}_p(3,8)$

corank-vector ρ_M ▶ assigns the difference $d - \mathrm{rank}(S)$ to any d-subset S of $[n]$, for a fixed matroid M of rank d on n elements

▶ The corank-vector ρ_M is contained in $\mathrm{Dr}(d,n)$, and also in $\mathrm{TGr}_p(d,n)$ if and only if M is realizable over an algebraically closed field of characteristic p. [Speyer 2005]

▶ $\mathrm{Dr}(3,8)$ is a non-pure non-simplicial nine-dimensional polyhedral fan with f-vector

$$(1, 15470, 642677, 8892898, 57394505, 194258750, 353149650, 324404880, 117594645, 113400)$$

[Herrmann, Joswig & Speyer 2012]

The following result is based on an explicit computation.

Theorem B (J.&S. 2015+)

The intersection of the relative interior of a cone C of $\mathrm{Dr}(3,8)$ and the tropical Grassmannian $\mathrm{TGr}_p(3,8)$ is trivial if $p = 0,3,5,7$ and a corank-vector ρ_M of a Fano matroid extension M is contained in the boundary of C.

note: The corank-vector ρ_M of an Fano extension induces a term ordering.

main steps in computation:

- compute a Gröbner basis of the Plücker ideal $I_{3,8} \otimes_{\mathbb{Z}} \mathbb{Z}_p$
- search for a polynomial f in this basis, s.t. the tropicalization of f has an unique minimum at ρ_M and this term is also minimal for all rays in C

The following complements Theorem B (again computational).

Theorem C (J.&S. 2015+)

Let $p = 0,3,5$ or 7. There are at least two relative interior points in the intersection of each cone of $\mathrm{Dr}(3,8)$ with $\mathrm{TGr}_p(3,8)$, that do not contain a corank-vector ρ_M of a Fano matroid extension M in the boundary.

main step:

- verify that the ideal $\mathrm{in}_\pi(I_{3,8})$ of initial forms is monomial-free via Gröbner bases and saturation

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