Matroidal Subdivisions, Dressians and Tropical Grassmannians

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Overview

1. Introduction to tropical linear spaces
2. Matroids from hypersimplex splits
3. Multi-splits of hypersimplices
4. Algorithms for tropical linear spaces
5. The degree of a tropical basis
6. Polyhedral computations over Puiseux fractions

The goal: A better understanding of tropical linear spaces
and their moduli spaces.

The tools: combinatorics, matroid theory, polyhedral geometry,
commutative algebra, computations and algorithms.

Influenced by the work of Gel'fand, Kapranov, Zelevinsky, Develin, Sturmfels, Speyer, Herrmann, Joswig, Rincón, Fink, Jensen, Santos, Dress, Edmonds, Fujishige, Murota, Hirai, Allamigeon, Benchimol, Gaubert, Mayr, Meyer, ...

Matroids

A (d, n)-matroid assigns a rank $\mathrm{rk}(S) \in \mathbb{Z}$ to each subset of $S \subset [n]$, s.t.

$$\mathrm{rk}(\emptyset) = 0 \qquad \text{and} \qquad \mathrm{rk}([n]) = d$$

$$\mathrm{rk}(S) \;\leq\; \mathrm{rk}(S \cup e) \;\leq\; \mathrm{rk}(S) + 1 \quad \text{for } e \in [n]$$

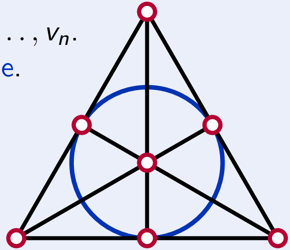
$$\mathrm{rk}(S) + \mathrm{rk}(S \cup f \cup g) \;\leq\; \mathrm{rk}(S \cup f) + \mathrm{rk}(S \cup g) \quad \text{for } f, g \in [n]$$

A d-set $S \subset [n]$ is a basis if $\mathrm{rk}(S) = d$.

Example

- The rank function of the uniform matroid is $\mathrm{rk}(S) = \min\{\# S, d\}$. All d-sets are bases in this case.
- $\mathrm{rk}(S) = \dim \mathrm{span}\{v_i : i \in S\}$ for vectors $v_1, \ldots, v_n$. The corresponding matroid is called realizable.
- Realizability depends on the underlying field.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$



Matroid Polytopes

The convex hull of the characteristic vectors of the bases of a matroid M is the matroid polytope. For the uniform matroid this is the hypersimplex

$$\Delta(d,n) \;=\; \left\{ x \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i = d \text{ and } x_i \geq 0 \right\} \;.$$

An outer description for the matroid polytope of M is

$$P_M \;=\; \left\{ x \in \Delta(d,n) \;\middle|\; \sum_{i \in F} x_i \leq \mathrm{rk}(F) \text{ for all subsets } F \text{ of } [n] \right\} \;.$$

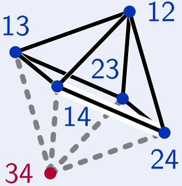
A set F is a flacet if the corresponding inequality is facet defining.

Example

The matroid of the columns of

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

has 5 bases and $\mathrm{rk}(34) = 1$

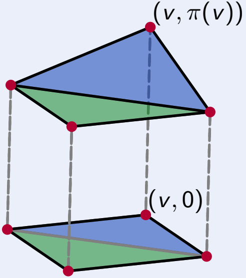


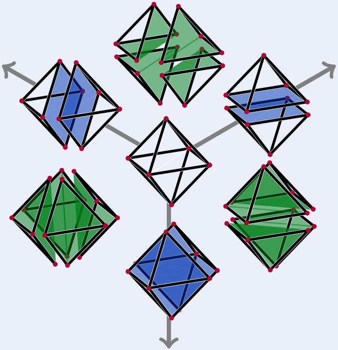
Regular Subdivisions and the Dressian

A subdivision is regular if it is induced by a lifting function. The lifting functions form a complete polyhedral fan, the secondary fan.

Liftings that decompose $\Delta(d, n)$ into matroid polytopes are called tropical Plücker vectors. The Dressian $\mathrm{Dr}(d, n)$ is the subfan of the secondary fan of those vectors. This is the moduli space of all tropical linear spaces.

Example





Tropical Grassmannians and Dressians

A tropical Grassmannian $\mathrm{TGr}_p(d, n)$ consists of the realizable tropical Plücker vectors. This $(dn - d^2 + 1)$-dim. fan depends on a char. of a field.

- Introduction of tropical varieties and the tropical Grassmannian. Study of the cases $\mathrm{TGr}_0(2, n)$ of phylogenetic trees and $\mathrm{TGr}_0(3, 6)$.
  (Speyer Sturmfels '04)
- Study of trop. lin. spaces and their f-vectors. (Speyer '05)
- Development of an algorithm to compute tropical varieties.
  (Bogart Jensen Speyer Sturmfels Thomas '07) (T. Markwig Ren '17$^+$)
- Introduction of the Dressian. Computation of $Dr(3, 7)$ and $TGr_p(3, 7)$. Including $\dim(\mathrm{Dr}(3, n)) \in \Theta(n^2)$. (Herrmann Jensen Joswig Sturmfels '09)
- Computation of $\mathrm{Dr}(3, 8)$ with a focus on the combinatorics of rays.
  (Herrmann Joswig Speyer '12)
- Local trop. lin. spaces, computing Bergman fans and trop. lin. spaces.
  (Rincón '13)(Hampe Joswig S. '17$^+$)
- The tropical Stiefel map and $\Delta_{d-1} \times \Delta_{n-d-1}$. (Fink Rincón '14)

Splits and Split Matroids

A split is a subdivision into exactly two maximal cells. Two split-hyperplanes are compatible if they do not intersect in the interior.

Lemma (Joswig, S. 2017)

- *The facets of a matroid polytope are either supported by hypersimplex splits or are hypersimplex facets.*
- *Two flacets F and G are compatible if and only if*

$$\#(F \cap G) + d \leq \mathrm{rk}(F) + \mathrm{rk}(G) \ .$$

We call a connected matroid a split matroid if its flacets form a compatible system of hypersimplex splits.

Split and Paving Matroids

A matroid is paving if its rank function satisfies

$$\mathrm{rk}(S) = \# S \quad \text{for all sets } S \text{ with } \# S \leq d-1\,.$$

It is conjectured that almost all matroids are paving.

Theorem (Joswig, S. 2017)

A connected (d, n)-matroid M is paving if and only if it is a split matroid such that each split flacet has rank $d-1$.

Remark: Split matroids are closed under dualization.
There is a excluded-minor characterization for the class of split matroids.

(Cameron Mayhew '17^{+})

The percentage of paving matroids among the isomorphism classes of all matroids of rank d on n elements

$d \backslash n$	4	5	6	7	8	9	10	11	12
2	57	46	43	38	36	33	32	30	29
3	50	31	24	21	21	30	52	78	91
4	100	40	22	17	34	77	–	–	–
5		100	33	14	12	63	–	–	–
6			100	29	10	14	–	–	–
7				100	25	7	17	–	–
8					100	22	5	19	–
9						100	20	4	16
10							100	18	3
11								100	17

This computation has been done with `polymake`.

It is based on data from Matsumoto, Moriyama, Imai and Bremner.

The percentage of split matroids among the isomorphism classes of all matroids of rank d on n elements

$d \backslash n$	4	5	6	7	8	9	10	11	12
2	100	100	100	100	100	100	100	100	100
3	100	100	89	75	60	52	61	80	91
4	100	100	100	75	60	82	–	–	–
5		100	100	100	60	82	–	–	–
6			100	100	100	52	–	–	–
7				100	100	100	61	–	–
8					100	100	100	80	–
9						100	100	100	91
10							100	100	100
11								100	100

This computation has been done with `polymake`.

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Corank Vectors and Corank Subdivisions

Given a matroid M, the corank-vector $\rho(M)$ is the height function that lifts the vertex of $\Delta(d, n)$ corresponding to S to height $d - \mathrm{rk}(S)$.

Lemma (Speyer 2005; Joswig, S. 2017)

The vector $\rho(M)$ corresponds to a tropicalization of a linear space if and only if M is realizable.

Speyer '05: The vector $\rho(M)$ is a tropical Plücker vector.

The polytope P_M occurs as a cell in the induced subdivision.

Fujishige '84, Feichtner Sturmfels '05:

This cell is maximal if and only if M is connected.

The Dimension of the Dressian

Theorem (Joswig, S. 2017)

The corank vector $\rho(M)$ of a split matroid M is contained in the interior of a simplicial cone in $Dr(d,n)$. Moreover,

$$\frac{1}{n}\binom{n}{d} - 1 \ \leq \ \dim \mathrm{Dr}(d,n) \ \leq \ \binom{n-2}{d-1} - 1 \ .$$

For fixed d the dimension of the Dressian is of order $\Theta(n^{d-1})$.

upper bound: Speyer's f-vector bound on the number of vertices of a tropical linear space.

lower bound: Knuth's construction for a stable set in the vertex-edge-graph of $\Delta(d,n)$ – a Johnson graph – gives a compatible set of vertex splits.

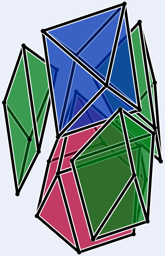
Rays of the Dressian

Let M be a connected split matroid and sf(M) the $(d+1, n+2)$-matroid, that is the series-extension at f of the free extension $M +_{[n]} f$.

Example

$(2, 6)$-matroid M with 3 split flacets. The corank subdivision of sf(M).

- $P_M \times P_{U_{1,2}}$ is a facet of $P_{\mathrm{sf}(M)}$.
- All facets are connected with $P_{\mathrm{sf}(M)}$.
- The dual graph is 2-connected.
- All cycles in the dual graph have length 3.



Rays of the Dressian

Let M be a connected split matroid and $\mathrm{sf}(M)$ the $(d+1, n+2)$-matroid, that is the series-extension at f of the free extension $M +_{[n]} f$.

Theorem (Joswig, S. 2017)

The corank vector of the series-free lift $\mathrm{sf}(M)$ *is a ray in the Dressian* $\mathrm{Dr}(d+1, n+2)$*. This ray corresponds to the tropicalization of a linear space if and only if* M *is realizable.*

Example

$(2,6)$-matroid M with 3 split flacets. The corank subdivision of $\mathrm{sf}(M)$.

- $P_M \times P_{U_{1,2}}$ is a facet of $P_{\mathrm{sf}(M)}$.
- All facets are connected with $P_{\mathrm{sf}(M)}$.
- The dual graph is 2-connected.
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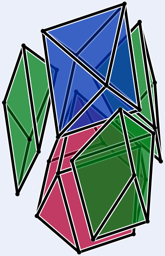
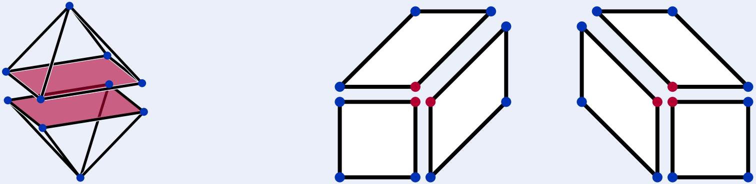


Multi-Splits

A multi-split is a subdivision into k maximal cells intersecting in a common $(k-1)$-codimensional cell. Splits are the special case of $k = 2$.

Example



Proposition (Herrmann 2009)

A multi-split with k maximal cells is a coarsest regular subdivision and induced by a polyhedral fan with k rays and k maximal cells.

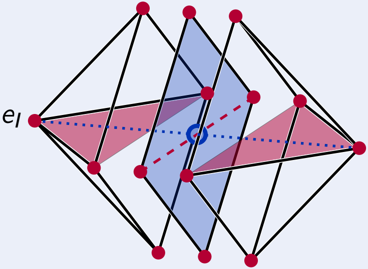
Multi-Splits of $\Delta(d, n)$

Theorem (S. 2017^+)

A multi-split of $\Delta(d, n)$ is the image of a multi-split of the vertex figure $\mathcal{F}(e_I) \cong \Delta_{d-1} \times \Delta_{n-d-1}$ under the Stiefel map if e_I is in the common cell. Moreover, a maximal cell in any multi-split of $\Delta(d, n)$ is the matroid polytope of a connected nested matroid. Each maximal cell determines the others uniquely.

Example

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Further Research

- Are almost all matroids split and all split matroids (sparse) paving?
- Is there a multi-split decomposition of matroid subdivisions?
- What is the relation to the intersection ring of Hampe, where nested matroids form bases?
- How to construct other rays of the Dressian or tropical Grassmannian.
- How to compute the facets of a secondary polytope, i.e., all regular coarsest subdivision?
- ...