Overview

Lecture 5: Spatial Diversity, MIMO Capacity
- SIMO, MISO, MIMO
- Degrees of freedom
- MIMO capacity

Lecture 6: MIMO Channel Modeling
Overview

Motivation:

- How does the multiplexing capability of MIMO channels depend on the physical environment?
- When can we gain (much) from MIMO?
- How do we have to design the system?

Physical Modeling

- Line-of-Sight Channels: SIMO

  - Free space without scattering and reflections.
  - Antenna separation $\Delta r, \lambda_c$, with carrier wavelength $\lambda_c$ and the normalized antenna separation $\Delta r; n_r$ receive antennas.
  - Distance between transmitter and $i$-th receive antenna: $d_i$
  - Continuous-time impulse between transmitter and $i$-th receive antenna:
    $$h_i(\tau) = a \cdot \delta(\tau - d_i/c)$$
  - Base-band model (assuming $d_i/c \ll 1/W$, signal BW $W$):
    $$h_i = a \cdot \exp\left(-j \frac{2\pi f_c d_i}{c}\right) = a \cdot \exp\left(-j \frac{2\pi d_i}{\lambda_c}\right)$$
  - SIMO model: $y = h \cdot x + w$, with $w \sim \mathcal{CN}(0, N_0 I)$
  - $h$: signal direction, spatial signature.
Physical Modeling
- Line-of-Sight Channels: SIMO
  - Paths are approx. parallel, i.e.,
    \[ d_i \approx d + (i-1)\Delta_r \cos(\phi) \]
  - Directional cosine
    \[ \Omega = \cos(\phi) \]
  - Spatial signature can be expressed as
    \[ h = a \cdot \exp \left( \frac{-j2\pi d}{\lambda_c} \right) \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_r \Omega) \\ \exp(-j2\pi2\Delta_r \Omega) \\ \vdots \\ \exp(-j2\pi(n_r - 1)\Delta_r \Omega) \end{bmatrix} \]
    \[ \rightarrow \text{Phased-array antenna.} \]
  - SIMO capacity (with MRC)
    \[ C = \log \left( 1 + \frac{P||h||^2}{N_0} \right) = \log \left( 1 + \frac{P a^2 n_r}{N_0} \right) \]
    \[ \rightarrow \text{Only power gain, no degree-of-freedom gain.} \]

Physical Modeling
- Line-of-Sight Channels: MISO
  - Similar to the SIMO case:
    \[ \Delta_t, \lambda_c, d_i, \phi, \Omega,... \]
  - MISO channel model:
    \[ y = h^*x + w, \]
    \[ \text{with } w \sim \mathcal{CN}(0, N_0). \]
  - Channel vector
    \[ h = a \cdot \exp \left( j \frac{2\pi d}{\lambda_c} \right) \begin{bmatrix} 1 \\ \exp(-j2\pi\Delta_t \Omega) \\ \exp(-j2\pi2\Delta_t \Omega) \\ \vdots \\ \exp(-j2\pi(n_t - 1)\Delta_t \Omega) \end{bmatrix} \]
  - Unit spatial signature in the directional cosine \( \Omega \):
    \[ e(\Omega) = 1/\sqrt{n} \cdot [1, \exp(-j2\pi\Delta_\Omega), \ldots, \exp(-j2\pi(n - 1)\Delta_\Omega)]^T \]
    \[ \rightarrow e_i(\Omega_i) \text{ and } e_r(\Omega_r) \text{ with } n_i, \Delta_i \text{ and } n_r, \Delta_r, \text{ respectively.} \]
Physical Modeling
- Line-of-Sight Channels: MIMO

• Linear transmit and receive array with \( n_t, \Delta_t \) and \( n_r, \Delta_r \).

• Gain between transmit antenna \( k \) and receive antenna \( i \)
  \[ h_{ik} = a \cdot \exp \left( -j \frac{2\pi d_{ik}}{\lambda_c} \right) \]

• Distance between transmit antenna \( k \) and receive antenna \( i \)
  \[ d_{ik} = d + (i - 1) \Delta_r \lambda_c \cos(\phi_r) - (k - 1) \Delta_t \lambda_c \cos(\phi_t) \]

• MIMO channel matrix (with \( \Omega_i = \cos(\phi_i) \) and \( \Omega_r = \cos(\phi_r) \))
  \[ H = a \sqrt{n_t n_r} \exp \left( -j \frac{2\pi d}{\lambda_c} \right) e_r(\Omega_r)e_t(\Omega_t)^* \]
  → \( H \) is a rank-1 matrix with singular value \( \lambda_1 = a \sqrt{n_t n_r} \)
  → Compare with SVD decomposition in Lecture 5:
  \[ H = \sum_{i=1}^{k} \lambda_i u_i v_i^* \]

Physical Modeling
- Line-of-Sight Channels: MIMO

• MIMO capacity
  \[ C = \log \left( 1 + \frac{P_d^2 n_t n_r}{N_0} \right) \]
  → Only power gain, no degree-of-freedom gain.
  - \( n_t = 1 \): power gain equals \( n_r \) → receive beamforming.
  - \( n_r = 1 \): power gain equals \( n_t \) → transmit beamforming.
  - General \( n_t, n_r \): power gain equals \( n_t \cdot n_r \)
    → Transmit and receive beamforming.

• Conclusion: In LOS environment, MIMO provides only a power gain
  but no degree-of-freedom gain.
Physical Modeling
– Geographically Separated Antennas at the Transmitter

Example/special case

- 2 distributed transmit antennas, attenuations $a_1, a_2$, angles of incidence $\phi_1, \phi_2$, negligible delay spread.
- Spatial signature ($n_r$ receive antennas)
  \[ h_k = a_k \sqrt{n_r} \exp \left( -j \frac{2\pi d_{1k}}{\lambda_c} \right) e_r(\Omega_{rk}) \]
- Channel matrix $H = [h_1, h_2]$

- $H$ has independent columns as long as ($\Omega_r = \cos(\phi_r)$)
  \[ \Omega_r = \Omega_2 - \Omega_1 \neq 0 \mod \frac{1}{\Delta_r} \]

$\to$ Two non-zero singular values $\lambda_1^2, \lambda_2^2$; i.e., two degrees of freedom.
$\to$ But $H$ can still be ill-conditioned!

Physical Modeling
– Geographically Separated Antennas at the Transmitter

- Conditioning of $H$ is determined by how the spatial signatures are aligned (with $L_r = n_r \Delta_r$):
  \[ |\cos(\theta)| = |f_r(\Omega_2 - \Omega_1)| = |e_r(\Omega_1) + e_r(\Omega_2)| = \frac{\sin(\pi L_r \Omega_r)}{n_r \sin(\pi L_r / n_r)} \]

- Example ($a_1 = a_2 = a$)
  \[ \lambda_1^2 = a^2 n_r (1 + |\cos(\theta)|) \]
  \[ \lambda_2^2 = a^2 n_r (1 - |\cos(\theta)|) \]
  \[ \Rightarrow \frac{\lambda_1}{\lambda_2} = \sqrt{1 + |\cos(\theta)|} \]

- $f_r(\Omega_r)$ is periodic with $n_r/L_r$.
- Maximum at $\Omega_r = 0$; $f_r(0) = 1$.
- $f_r(\Omega_r) = 0$ at $\Omega_r = k/L_r$ with $k = 1, \ldots, n_r - 1$.
- Resolvability $1/L_r$,
  if $\Omega_r \ll 1/L_r$, then the signals from the two antennas cannot be resolved.
Physical Modeling
- Geographically Separated Antennas at the Transmitter

Beamforming pattern

- Assumption: signal arrives with angle $\phi_0$; receive beamforming vector $e_r(\cos(\phi_0))$.
- A signal from any other direction $\phi$ will be attenuated by a factor
  
  $|e_r(\cos(\phi_0))e_r(\cos(\phi))| = |f_r(\cos(\phi) - \cos(\phi_0))|$

- Beamforming pattern
  
  \( (\phi, |f_r(\cos(\phi) - \cos(\phi_0))| ) \)

- Main lobes around $\phi_0$ and any angle $\phi$ for which $\cos(\phi) = \cos(\phi_0)$.
  
  → In a similar way, separated receive antennas can be treated.

Physical Modeling
- LOS Plus One Reflected Path

- Direct path: $\phi_{t1}$, $\Omega_{t1}$, $d^{(1)}$, and $a_1$.
- Reflected path: $\phi_{t2}$, $\Omega_{t2}$, $d^{(2)}$, and $a_2$.

- Channel model follows from signal superposition
  
  \[ H = a_{t1}^b e_t(\Omega_{t1})e_r(\Omega_{t1})^* + a_{t2}^b e_t(\Omega_{t2})e_r(\Omega_{t2})^* \]

  with
  
  \[ a_{t i}^b = a_i \sqrt{\frac{m}{n}} \exp \left( -j \frac{2 \pi d^{(i)}}{\lambda_c} \right) \]

  → $H$ has rank 2 as long as
  
  $\Omega_{t1} \neq \Omega_{t2} \mod \frac{1}{\Delta_t}$ and $\Omega_{t1} \neq \Omega_{t2} \mod \frac{1}{\Delta_r}$.

  → $H$ is well conditioned if the angular separations $|\Omega_t|, |\Omega_r|$ at the transmit/receive array are of the same order or larger than $1/L_{t,r}$. 

Notes
Physical Modeling
- LÖS Plus One Reflected Path

- Direct path:
  $\phi_1$, $\Omega_1$, $d(1)$, and $a_1$.

- Reflected path:
  $\phi_2$, $\Omega_2$, $d(2)$, and $a_2$.

(D. Tse and P. Viswanath, Fundamentals of Wireless Communications.)

$H$ can be rewritten as $H = H''H'$, with

$$H'' = \begin{bmatrix} a_1 e_1(\Omega_1) \\ a_2 e_1(\Omega_2) \end{bmatrix} \quad \text{and} \quad H' = \begin{bmatrix} e_t^*(\Omega_1) \\ e_t^*(\Omega_2) \end{bmatrix}$$

$\Rightarrow$ Two imaginary receivers at points A and B (virtual relays).

- Since the points A and B are geographically widely separated, $H'$ and $H''$ have rank 2 and hence $H$ has rank 2 as well.

- Furthermore, if $H'$ and $H''$ are well-conditioned, $H$ will be well-conditioned as well.

$\Rightarrow$ Multipath fading can be viewed as an advantage which can be exploited!

Physical Modeling
- LÖS Plus One Reflected Path

(D. Tse and P. Viswanath, Fundamentals of Wireless Communications.)

- Significant angular separation is required at both the transmitter and the receiver to obtain a well-conditioned matrix $H$.

- If the reflectors are close to the receiver (downlink), we have a small angular separation $\Rightarrow$ not very well-conditioned matrix $H$.

- Similar, if the reflectors are close to the transmitter (uplink).

$\Rightarrow$ Size of an antenna array at a base station will have to be many wavelengths to be able to exploit the spatial multiplexing effect.
Modeling of MIMO Fading Channels

- General Concept

- Antenna lengths $L_t, L_r$ limit the resolvability of the transmit and receive antenna in the angular domain.

- Sample the angular domain at fixed angular spacings of $1/L_t$ at the transmitter and $1/L_r$ at the receiver.

- Represent the channel (the multiple paths) in terms of these input and output coordinates.

- The $(k, l)$-th channel gain follows as the aggregation of all paths whose transmit and receive directional cosines lie in a $(1/L_t \times 1/L_r)$ bin around the point $(k/L_t, l/L_r)$.

(D. Tse and P. Viswanath, Fundamentals of Wireless Communications)

Note, if $\Omega_t, \Omega_r \ll 1/L_t, 1/L_r$, the paths cannot be separated.
Modeling of MIMO Fading Channels
– Angular Domain Representation (ADR)

- Orthonormal basis for the received signal space ($n_r$ basis vectors)

$$S_r = \{ e_r(0), e_r(\frac{1}{L_r}), \ldots, e_r(n_r - 1) \}$$

→ Orthogonality follows directly from the properties of $f_r(\Omega)$.

- Orthonormal basis for the transmitted signal space ($n_t$ basis vectors)

$$S_t = \{ e_t(0), e_t(\frac{1}{L_t}), \ldots, e_t(n_t - 1) \}$$

→ Orthogonality follows directly from the properties of $f_t(\Omega)$.

- Orthonormal bases provide a very simple (but approximate) decomposition of the total received/transmitted signal up to a resolution $1/L_r, 1/L_t$.

Examples: Receive beamform patterns of the angular basis vectors in $S_r$

- (a) Critically spaced ($\Delta_r = 1/2$), each basis vector has a single pair of main lobes.

- (b) Sparsely spaced ($\Delta_r > 1/2$), some of the basis vectors have more than one pair of main lobes.

- (c) Densely spaced ($\Delta_r < 1/2$), some of the basis vectors have no pair of main lobes.

(D. Tse and P. Viswanath, Fundamentals of Wireless Communications.)
Modeling of MIMO Fading Channels
– AdR of MIMO Channels

(Assumption: critically spaced antennas)

• Observation: The vectors in $S_t$ and $S_r$ form unitary matrices $U_t$ and $U_r$ with dimensions $(n_t \times n_t)$ and $(n_r \times n_r)$, respectively. ($\rightarrow$ IDFT matrices!)

• With $y^a = U_t^* x$ and $y^r = H^* x + w^r$, we get $y^a = H^a x^a + w^a$, with $w^a \sim CN(0, N_0 I_{n_r})$.

• Furthermore, with $H = \sum_i a^a_i e_i(\Omega_i) e_i(\Omega_i)^*$, we get

$$h_{ij}^a = e_r(k/L_r) H e_t(l/L_t)$$

$$= \sum_i a^a_i e_r(k/L_r)^* e_r(\Omega_i)^* e_t(l/L_t)$$

(1)

(2)

• The terms (1) and (2) are significant for the $i-th$ path if

$$|\Omega_{ij} - k/L_r| < 1/L_r$$

and

$$|\Omega_{ij} - k/L_t| < 1/L_t$$

$\rightarrow$ Projections on the basis vectors in $S_t, S_r$.

$^3$The superscript "a" denotes angular domain quantities.

Modeling of MIMO Fading Channels
– Statistical Modeling in the Angular Domain

• Let $T_l$ and $R_k$ be the sets of physical paths which have most energy in directions of $e_t(l/L_t)$ and $e_r(k/L_r)$.

• $h_{ij}^a$ corresponds to the aggregated gains $a_i^a$ of paths which lie in $R_k \cap T_l$.

• Independence and time variation

  • Gains of the physical paths $a_i^a[m]$ are independent.

  $\Rightarrow$ The path gains $h_{ij}^a[m]$ are independent across $m$.

  • The angles $\{\phi_{ij}^a[m]\}_m$ and $\{\phi_{ij}^r[m]\}_m$ evolve slower than $a_i^a[m]$.

  $\Rightarrow$ The physical paths do not move from one angular bin to another. $\Rightarrow$ The path gains $h_{ij}^a[m]$ are independent across $k$ and $l$.

• If there are many paths in an angular bin $\Rightarrow$ Central Limit Theorem $\Rightarrow h_{ij}^a[m]$ can be modeled as complex circular symmetric Gaussian.

• If there are no paths in an angular bin $\Rightarrow h_{ij}^a[m] \approx 0$.

• Since $U_t$ and $U_r$ are unitary matrices, the matrix $H$ has the same i.i.d. Gaussian distribution as $H^r$. 
Modeling of MIMO Fading Channels

- Degrees of Freedom and Diversity

Degrees of freedom

- Based on the derived statistical model, we get the following result: with probability 1, the rank of the random matrix $H^a$ is given by

$$\text{rank}(H^a) = \min\{ \text{number of non-zero rows}, \text{number of non-zero columns} \}.$$ 

- The number of non-zero rows and columns depends on two factors:
  - Amount of scattering and reflection; the more scattering and reflection, the larger the number of non-zero entries in $H^a$.
  - Lengths $L_r$ and $L_t$; for small $L_r$, $L_t$, many physical paths are mapped into the same angular bin; with higher resolution, more paths can be represented.

Diversity: The diversity is given by the number of non-zero entries in $H^a$.

Example: Same number of degrees of freedom but different diversity.

(D. Tse and P. Viswanath, Fundamentals of Wireless Communications.)
Modeling of MIMO Fading Channels

- Antenna Spacing

So far: critically spaced antennas with $\Delta_r = 1/2$.

- One-to-one correspondence between the angular windows and the resolvable bins.

Setup 1: vary the number of antennas for a fixed array length $L_{r,t}$.

- Sparsely spaced case ($\Delta_r > 0.5$)
  - Beamforming patterns of some basis vectors have multiple main lobes.
  - Different paths with different directions are mapped onto the same basis vector.
  - Resolution of the antenna array, number of degrees of freedom, and diversity are reduced.

- Densely spaced case ($\Delta_r < 0.5$)
  - There are basis vectors with no main lobes which do not contribute to the resolvability.
  - Adds zero rows and columns to $H_a$ and creates correlation in $H$.

Setup 2: vary the antenna separation for a fixed number of antennas.

- Rich scattering: number of non-zero rows in $H_a$ is already $n_r$; i.e., no improvement possible.

- Clustered scattering: scattered signal can be received in more bins; i.e., increasing number of degrees of freedom.