Optimal input design for nonlinear dynamical systems: a graph-theory approach

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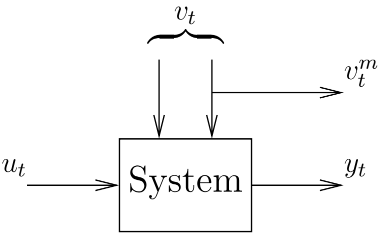
Seminar Uppsala university
January 16, 2015

System identification



- System identification: Modeling based on input-output data.
- Basic entities: data set, model structure, identification method.

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System identification

The maximum likelihood method:

$$\hat{\theta}_{n_{\text{seq}}} = \arg\max_{\theta \in \Theta} l_\theta(y_{1:n_{\text{seq}}})$$

where

$$l_\theta(y_{1:n_{\text{seq}}}) := \log p_\theta(y_{1:n_{\text{seq}}})$$

As $n_{\text{seq}} \to \infty$, we have:

- $\hat{\theta}_{n_{\text{seq}}} \to \theta_0$
- $\sqrt{n_{\text{seq}}}\left(\hat{\theta}_{n_{\text{seq}}} - \theta_0\right) \to$ Normal with zero mean and covariance $\{\mathcal{I}_F^e\}^{-1}$

System identification

- $\sqrt{n_{\text{seq}}}\left(\hat{\theta}_{n_{\text{seq}}} - \theta_0\right) \to$ Normal with zero mean and covariance $\{\mathcal{I}_F^e\}^{-1}$

where

$$\mathcal{I}_F^e := \mathbf{E}\left\{ \frac{\partial}{\partial\theta} l_\theta(y_{1:n_{\text{seq}}})\Big|_{\theta=\theta_0} \frac{\partial}{\partial\theta^\top} l_\theta(y_{1:n_{\text{seq}}})\Big|_{\theta=\theta_0} \Big| u_{1:n_{\text{seq}}} \right\}$$

Different $u_{1:n_{\text{seq}}}$s $\Rightarrow$ different $\mathcal{I}_F^e$s.

Covariance matrix of $\sqrt{n_{\text{seq}}}\left(\hat{\theta}_{n_{\text{seq}}} - \theta_0\right)$ affected by $u_{1:n_{\text{seq}}}$!

Input design for dynamic systems

- Input design: Maximize information from an experiment.
- Existing methods: focused on linear systems.
- Recent developments for nonlinear systems (Hjalmarsson 2007, Larsson 2010, Gopaluni 2011, De Cock-Gevers-Schoukens 2013, Forgione-Bombois-Van den Hof-Hjalmarsson 2014).

Input design for dynamic systems

Challenges for nonlinear systems:

- Problem complexity (usually non-convex).
- Model restrictions.
- Input restrictions.

How could we overcome these limitations?

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Summary

1. We present a method for input design for dynamic systems.

2. The method is also suitable for nonlinear systems.

Outline

Problem formulation for output-error models

Input design based on graph theory

Extension to nonlinear SSM

Closed-loop application oriented input design

Conclusions and future work

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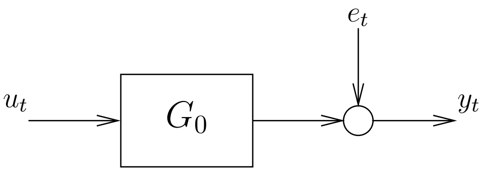
Problem formulation for output-error models



G_0 is a system, where:

- e_t: white noise (variance λ_e)
- u_t: input
- y_t: system output

Goal: Design

$$u_{1:n_{\mathrm{seq}}} := (u_1, \ldots, u_{n_{\mathrm{seq}}})$$

as a realization of a stationary process *maximizing* $\mathcal{I}_F$.

Problem formulation for output-error models

Here,

$$\mathcal{I}_F := \frac{1}{\lambda_e}\mathbf{E}\left\{\sum_{t=1}^{n_{\text{seq}}} \psi_t^{\theta_0}(u_t)\psi_t^{\theta_0}(u_t)^\top\right\}$$

$$= \frac{1}{\lambda_e}\int \sum_{t=1}^{n_{\text{seq}}} \psi_t^{\theta_0}(u_t)\psi_t^{\theta_0}(u_t)^\top \, dP(u_{1:n_{\text{seq}}})$$

$$\psi_t^{\theta_0}(u_t) := \nabla_\theta \hat{y}_t(u_t)\big|_{\theta=\theta_0}$$

$$\hat{y}_t(u_t) := G(u_t;\, \theta)$$

Design $u_{1:n_{\text{seq}}} \in \mathbb{R}^{n_{\text{seq}}} \Leftrightarrow$ Design $P(u_{1:n_{\text{seq}}}) \in \mathcal{P}$.

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Problem formulation for output-error models

Here,

$$\begin{aligned}\mathcal{I}_F &:= \frac{1}{\lambda_e}\mathbf{E}\left\{\sum_{t=1}^{n_{\text{seq}}} \psi_t^{\theta_0}(u_t)\psi_t^{\theta_0}(u_t)^\top\right\} \\ &= \frac{1}{\lambda_e}\int \sum_{t=1}^{n_{\text{seq}}} \psi_t^{\theta_0}(u_t)\psi_t^{\theta_0}(u_t)^\top \, dP(u_{1:n_{\text{seq}}})\end{aligned}$$

Assumption

$u_t \in \mathcal{C}$ *($\mathcal{C}$ finite set)*

$$\mathcal{I}_F = \frac{1}{\lambda_e}\sum_{u_{1:n_{\text{seq}}}\in\mathcal{C}^{n_{\text{seq}}}}\sum_{t=1}^{n_{\text{seq}}} \psi_t^{\theta_0}(u_t)\psi_t^{\theta_0}(u_t)^\top p(u_{1:n_{\text{seq}}})$$

Problem formulation for output-error models

Characterizing $p(u_{1:n_{\text{seq}}}) \in \mathcal{P}_{\mathcal{C}}$:

- p nonnegative,
- $\sum p(\mathbf{x}) = 1$,
- p is shift invariant.

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Problem formulation for output-error models

Problem

Design $u_{1:n_{\text{seq}}}^{\text{opt}} \in \mathcal{C}^{n_{\text{seq}}}$ as a realization from $p^{\text{opt}}(u_{1:n_{\text{seq}}})$, where

$$p^{\text{opt}}(u_{1:n_{\text{seq}}}) := \arg\max_{p \in \mathcal{P}_{\mathcal{C}}} h(\mathcal{I}_F(p))$$

where $h : \mathbb{R}^{n_\theta \times n_\theta} \to \mathbb{R}$ is a matrix concave function, and

$$\mathcal{I}_F(p) = \frac{1}{\lambda_e} \sum_{u_{1:n_{\text{seq}}} \in \mathcal{C}^{n_{\text{seq}}}} \sum_{t=1}^{n_{\text{seq}}} \psi_t^{\theta_0}(u_t) \psi_t^{\theta_0}(u_t)^\top p(u_{1:n_{\text{seq}}})$$

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Problem formulation for output-error models

Input design based on graph theory

Extension to nonlinear SSM

Closed-loop application oriented input design

Conclusions and future work

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Input design problem

Problem

Design $u_{1:n_{\text{seq}}}^{\text{opt}} \in \mathcal{C}^{n_{\text{seq}}}$ as a realization from $p^{\text{opt}}(u_{1:n_{\text{seq}}})$, where

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Issues:

1. $\mathcal{I}_F(p)$ requires a sum of n_{seq}-dimensional terms (n_{seq} large).
2. How could we represent an element in $\mathcal{P}_{\mathcal{C}}$?

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Input design problem

Solving the issues:

1. $\mathcal{I}_F(p)$ requires a sum of n_{seq}-dimensional terms (n_{seq} large).

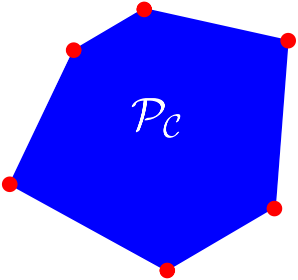
Assumption

$u_{1:n_{\text{seq}}}$ is a realization of a stationary process with memory n_m ($n_m < n_{\text{seq}}$).

$\Rightarrow \mathcal{I}_F(p)$ requires a sum of n_m-dimensional terms.

Minimum n_m: related with the memory of the system.

Input design problem



Solving the issues:

2. How could we represent an element in $\mathcal{P}_{\mathcal{C}}$?

$\mathcal{P}_{\mathcal{C}}$ is a polyhedron.

$\mathcal{V}_{\mathcal{P}_{\mathcal{C}}}$: Set of extreme points of $\mathcal{P}_{\mathcal{C}}$.

$\Rightarrow \mathcal{P}_{\mathcal{C}}$ can be described as a convex combination of $\mathcal{V}_{\mathcal{P}_{\mathcal{C}}}$.

The elements in $\mathcal{V}_{\mathcal{P}_{\mathcal{C}}}$ can be found by using Graph theory!

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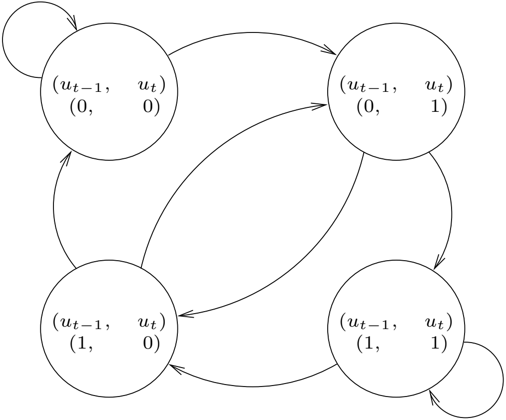
Graph theory in input design

Example: de Bruijn graph, $\mathcal{C} := \{0, 1\}$, $n_m := 2$.



Elements in $\mathcal{V}_{\mathcal{P}_\mathcal{C}}$ $\Leftrightarrow$ Prime cycles in $\mathcal{G}_{\mathcal{C}^{n_m}}$

Prime cycles in $\mathcal{G}_{\mathcal{C}^{n_m}}$ $\Leftrightarrow$ Elementary cycles in $\mathcal{G}_{\mathcal{C}^{(n_m-1)}}$

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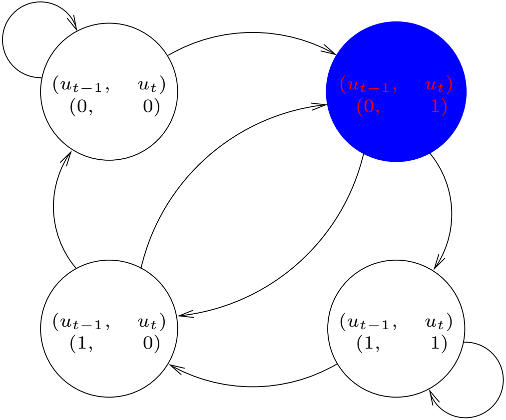

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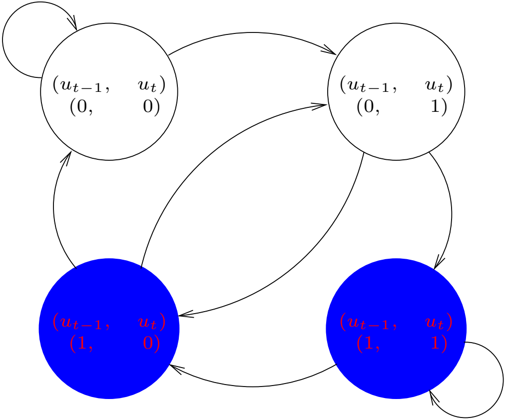

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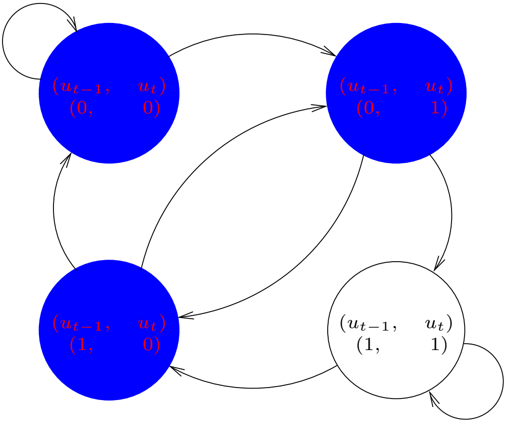

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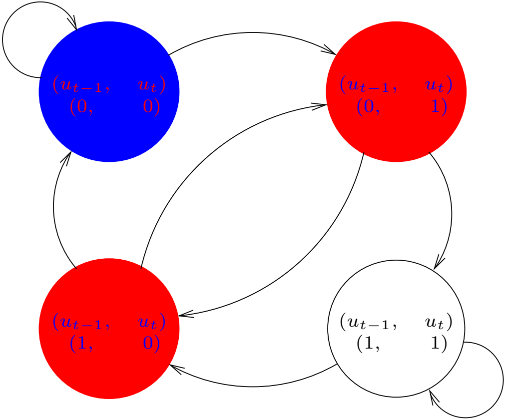

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Graph theory in input design

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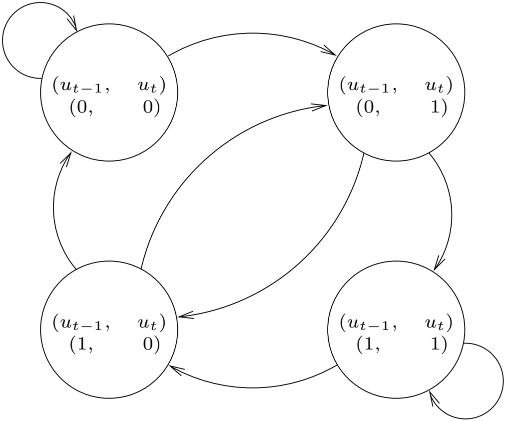

There are algorithms to find elementary cycles (Johnson 1975, Tarjan 1972).

Graph theory in input design

Once $v_i \in \mathcal{V}_{\mathcal{P}_\mathcal{C}}$ is known

$\Rightarrow$ The distribution for each v_i is known.

$\Rightarrow$ An input signal $\{u_t^i\}_{t=0}^{t=N}$ can be drawn from v_i.

Therefore,

$$\mathcal{I}_F^{(i)} := \frac{1}{\lambda_e} \sum_{u_{1:n_m} \in \mathcal{C}^{n_m}} \sum_{t=1}^{n_m} \psi_t^{\theta_0}(u_t) \psi_t^{\theta_0}(u_t)^\top v_i(u_{1:n_m})$$
$$\approx \frac{1}{\lambda_e N} \sum_{t=1}^{N} \psi_t^{\theta_0}(u_t) \psi_t^{\theta_0}(u_t)^\top$$

for all $v_i \in \mathcal{V}_{\mathcal{P}_\mathcal{C}}$.

Graph theory in input design

Therefore,

$$\mathcal{I}_F^{(i)} := \frac{1}{\lambda_e} \sum_{u_{1:n_m} \in \mathcal{C}^{n_m}} \sum_{t=1}^{n_m} \psi_t^{\theta_0}(u_t) \psi_t^{\theta_0}(u_t)^\top v_i(u_{1:n_m})$$
$$\approx \frac{1}{\lambda_e\, N} \sum_{t=1}^{N} \psi_t^{\theta_0}(u_t) \psi_t^{\theta_0}(u_t)^\top$$

for all $v_i \in \mathcal{V}_{\mathcal{P}_\mathcal{C}}$.

The sum is approximated by Monte-Carlo!

Input design based on graph theory

To design an experiment in $\mathcal{C}^{n_m}$:

1. Compute all the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$.
2. Generate the input signals $\{u_t^i\}_{t=0}^{t=N}$ from the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$, for each $i \in \{1, \dots, n_{\mathcal{V}}\}$.
3. For each $i \in \{1, \dots, n_{\mathcal{V}}\}$, approximate $\mathcal{I}_F^{(i)}$ by using

$$\mathcal{I}_F^{(i)} \approx \frac{1}{\lambda_e N} \sum_{t=1}^{N} \psi_t^{\theta_0}(u_t) \psi_t^{\theta_0}(u_t)^\top$$

Input design based on graph theory

To design an experiment in $\mathcal{C}^{n_m}$:

4. Define $\gamma := \{\alpha_1, \dots, \alpha_{n_\mathcal{V}}\} \in \mathbb{R}^{n_\mathcal{V}}$.
   Solve
   $$\gamma^{\text{opt}} := \arg \max_{\gamma \in \mathbb{R}^{n_\mathcal{V}}} h(\mathcal{I}_F^{\text{app}}(\gamma))$$
   where
   $$\mathcal{I}_F^{\text{app}}(\gamma) := \sum_{i=1}^{n_\mathcal{V}} \alpha_i \, \mathcal{I}_F^{(i)}$$
   $$\sum_{i=1}^{n_\mathcal{V}} \alpha_i = 1$$
   $$\alpha_i \geq 0 \,, \text{ for all } i \in \{1, \dots, n_\mathcal{V}\}$$

Input design based on graph theory

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To design an experiment in $\mathcal{C}^{n_m}$:

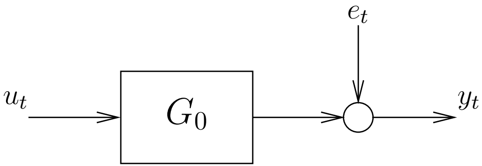
5. The optimal pmf p^{opt} is given by

$$p^{\text{opt}} = \sum_{i=1}^{n_{\mathcal{V}}} \alpha_i^{\text{opt}} \, v_i$$

6. Sample $u_{1:n_{\text{seq}}}$ from p^{opt} using Markov chains.

$\mathcal{I}_F^{\text{app}}(\gamma)$ linear in the decision variables $\Rightarrow$ The problem is convex!

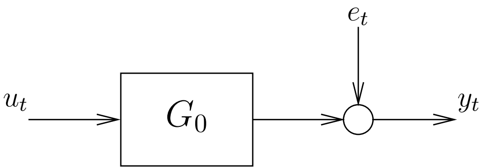
Example I



$$G(u_t;\theta) = \begin{cases} x_{t+1} = \dfrac{1}{\theta_1 + x_t^2} + u_t \\ y_t = \theta_2\, x_t^2 + e_t \\ x_1 = 0 \end{cases}$$

with $\theta = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^\top = \theta_0 = \begin{bmatrix}0.8 & 2\end{bmatrix}^\top$.
e_t: white noise, Gaussian, zero mean, variance $\lambda_e = 1$.

Example I



$$G(u_t;\theta) = \begin{cases} x_{t+1} = \dfrac{1}{\theta_1 + x_t^2} + u_t \\ y_t = \theta_2\, x_t^2 + e_t \\ x_1 = 0 \end{cases}$$

with $\theta = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^\top = \theta_0 = \begin{bmatrix}0.8 & 2\end{bmatrix}^\top$.
We consider $h(\cdot) = \log\det(\cdot)$, and $n_{\text{seq}} = 10^4$.

Example I

$$G(u_t;\theta) = \begin{cases} x_{t+1} = \dfrac{1}{\theta_1 + x_t^2} + u_t \\ y_t = \theta_2\, x_t^2 + e_t \\ x_1 = 0 \end{cases}$$

Results:

$h(\mathcal{I}_F)$	Case 1	Case 2	Case 3	Binary
$\log\{\det(\mathcal{I}_F)\}$	3.82	4.50	4.48	3.47

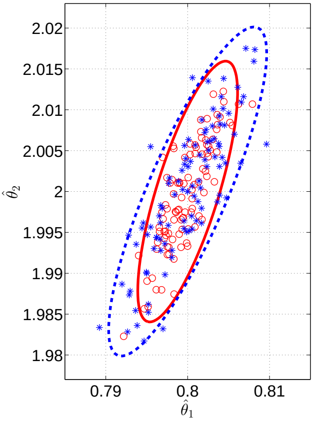
Case 1: $n_m = 2$, $\mathcal{C} = \{-1,\, 0,\, 1\}$

Case 2: $n_m = 1$, $\mathcal{C} = \{-1,\, -1/3,\, 1/3,\, 1\}$

Case 3: $n_m = 1$, $\mathcal{C} = \{-1,\, -0.5,\, 0,\, 0.5,\, 1\}$

Example I

Results ($95\,\%$ confidence ellipsoids):



Red: Case 2; **Blue:** Binary input.

Problem formulation for output-error models

Input design based on graph theory

Extension to nonlinear SSM

Closed-loop application oriented input design

Conclusions and future work

Extension to nonlinear SSM

Nonlinear state space model:

$$x_0 \sim \mu(x_0)$$
$$x_t | x_{t-1} \sim f_\theta(x_t | x_{t-1}, u_{t-1})$$
$$y_t | x_t \sim g_\theta(y_t | x_t, u_t)$$

where $\theta \in \Theta$.

- f_θ, g_θ, μ: pdfs
- x_t: states
- u_t: input
- y_t: system output

Goal: Design

$$u_{1:n_{\text{seq}}} := (u_1, \ldots, u_{n_{\text{seq}}})$$

as a realization of a stationary process *maximizing* $\mathcal{I}_F$.

Extension to nonlinear SSM

Here,

$$\mathcal{I}_F = \mathbf{E}\left\{\mathcal{S}(\theta_0)\mathcal{S}^\top(\theta_0)\right\}$$
$$\mathcal{S}(\theta_0) = \nabla_\theta \log p_\theta(y_{1:n_{\text{seq}}}|u_{1:n_{\text{seq}}})\big|_{\theta=\theta_0}$$

Design $u_{1:n_{\text{seq}}} \in \mathbb{R}^{n_{\text{seq}}} \Leftrightarrow$ Design $P(u_{1:n_{\text{seq}}}) \in \mathcal{P}$.

Extension to nonlinear SSM

Fisher's identity:

$$\nabla_\theta \log p_\theta(y_{1:T}|u_{1:T}) = \mathbf{E}\left\{\nabla_\theta \log p_\theta(x_{1:T}, y_{1:T}|u_{1:T})\middle|y_{1:T}, u_{1:T}\right\}$$

$$\nabla_\theta \log p_\theta(y_{1:T}|u_{1:T}) = \sum_{t=1}^{T} \int_{\mathcal{X}^2} \xi_\theta(x_{t-1:t},\, u_t) p_\theta(x_{t-1:t}|y_{1:T}) \mathrm{d}x_{t-1:t}$$

with

$$\xi_\theta(x_{t-1:t},\, u_t) = \nabla_\theta \Big[\log f_\theta(x_t|x_{t-1}, u_{t-1}) + \log g_\theta(y_t|x_t, u_t) \Big]$$

Assumption

$u_t \in \mathcal{C}$ *($\mathcal{C}$ finite set)*

Extension to nonlinear SSM

Recall $\mathcal{P}_{\mathcal{C}}$:

- p nonnegative,
- $\sum p(\mathbf{x}) = 1$,
- p is shift invariant.

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Extension to nonlinear SSM

Problem

Design $u^{\text{opt}}_{1:n_{\text{seq}}} \in \mathcal{C}^{n_{\text{seq}}}$ as a realization from $p^{\text{opt}}(u_{1:n_{\text{seq}}})$, where

$$p^{\text{opt}}(u_{1:n_{\text{seq}}}) := \arg\max_{p \in \mathcal{P}_{\mathcal{C}}} h(\mathcal{I}_F(p))$$

where $h : \mathbb{R}^{n_\theta \times n_\theta} \to \mathbb{R}$ is a matrix concave function, and

$$\mathcal{I}_F(p) = \mathbf{E}\left\{\mathcal{S}(\theta_0)\mathcal{S}^\top(\theta_0)\right\}$$

Input design problem for nonlinear SSM

Problem

Design $u_{1:n_{\text{seq}}}^{\text{opt}} \in \mathcal{C}^{n_{\text{seq}}}$ as a realization from $p^{\text{opt}}(u_{1:n_{\text{seq}}})$, where

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Issues:

1. How could we represent an element in $\mathcal{P}_{\mathcal{C}}$? $\Rightarrow$ Use the graph theory approach!
2. How could we compute $\mathcal{I}_F(p)$?

Input design based on graph theory (revisited)

To design an experiment in $\mathcal{C}^{n_m}$:

1. Compute all the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$.
2. Generate the input signals $\{u_t^i\}_{t=0}^{t=N}$ from the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$, for each $i \in \{1, \ldots, n_{\mathcal{V}}\}$.
3. **For each $i \in \{1, \ldots, n_{\mathcal{V}}\}$, approximate $\mathcal{I}_F^{(i)}$ by using**

$$\mathcal{I}_F^{(i)} := \mathbf{E}_{v_i(u_{1:n_m})}\left\{\mathcal{S}(\theta_0)\mathcal{S}^\top(\theta_0)\right\}$$

$\approx$ **(new expression required!)**

Estimating $\mathcal{I}_F$

Approximate $\mathcal{I}_F$ as

$$\hat{\mathcal{I}}_F := \frac{1}{M} \sum_{m=1}^{M} \mathcal{S}_m(\theta_0)\mathcal{S}_m^\top(\theta_0)$$

- Difficulty: $\mathcal{S}_m(\theta_0)$ is not available.
- Solution: Estimate $\mathcal{S}_m(\theta_0)$ using particle methods!

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Particle methods to estimate $\mathcal{S}_m(\theta_0)$

- Goal: Approximate $\{p_\theta(x_{1:t}|y_{1:t})\}_{t\geq 1}$.
- $\{x_{1:t}^{(i)}, w_t^{(i)}\}_{i=1}^N$: Particle system.
- Approach: Auxiliary particle filter + Fixed-lag smoother.

Particle methods to estimate $\mathcal{S}_m(\theta_0)$

Estimate $\mathcal{S}_m(\theta_0)$ as

$$\hat{S}_m(\theta_0) := \sum_{t=1}^{T}\sum_{i=1}^{N} w_{\kappa_t}^{(i)} \xi_{\theta_0}(x_{t-1}^{a_{\kappa_t,t-1}^{(i)}},\, x_t^{a_{\kappa_t,t}^{(i)}},\, u_t)$$

where

$$\xi_\theta(x_{t-1:t},\, u_t) = \nabla_\theta\Big[\log f_\theta(x_t|x_{t-1}, u_{t-1}) + \log g_\theta(y_t|x_t, u_t)\Big]$$

Input design based on graph theory (revisited)

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1. Compute all the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$.
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3. **For each $i \in \{1, \ldots, n_\mathcal{V}\}$, approximate $\mathcal{I}_F^{(i)}$ by using**

$$\mathcal{I}_F^{(i)} := \mathbf{E}_{v_i(u_{1:n_m})} \left\{ \mathcal{S}(\theta_0) \mathcal{S}^\top(\theta_0) \right\}$$
$$\approx \frac{1}{M} \sum_{m=1}^{M} \hat{\mathcal{S}}_m(\theta_0) \hat{\mathcal{S}}_m^\top(\theta_0)$$

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Input design based on graph theory (revisited)

To design an experiment in $\mathcal{C}^{n_m}$:

4. Define $\gamma := \{\alpha_1, \ldots, \alpha_{n_\mathcal{V}}\} \in \mathbb{R}^{n_\mathcal{V}}$.
   For $k \in \{1, \ldots, K\}$**, solve**

$$\gamma^{\mathrm{opt},k} := \arg \max_{\gamma \in \mathbb{R}^{n_\mathcal{V}}} h(\mathcal{I}_F^{\mathrm{app},k}(\gamma_k))$$

   where

$$\mathcal{I}_F^{\mathrm{app},k}(\gamma_k) := \sum_{i=1}^{n_\mathcal{V}} \alpha_{i,k}\, \mathcal{I}_F^{(i),k}$$

$$\sum_{i=1}^{n_\mathcal{V}} \alpha_{i,k} = 1$$

$$\alpha_{i,k} \geq 0\,, \text{ for all } i \in \{1, \ldots, n_\mathcal{V}\}$$

   Compute $\overline{\gamma}$ **as the sample mean of** $\{\gamma^{\mathrm{opt},k}\}_{k=1}^{K}$**.**

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Input design based on graph theory (revisited)

To design an experiment in $\mathcal{C}^{n_m}$:

5. The optimal pmf p^{opt} is given by

$$p^{\text{opt}} = \sum_{i=1}^{n_{\mathcal{V}}} \overline{\alpha}_i^{\text{opt}} \, v_i$$

6. Sample $u_{1:n_{\text{seq}}}$ from p^{opt} using Markov chains.

$\mathcal{I}_F^{\text{app}}(\gamma)$ linear in the decision variables $\Rightarrow$ The problem is convex!

Example II

Nonlinear state space model:

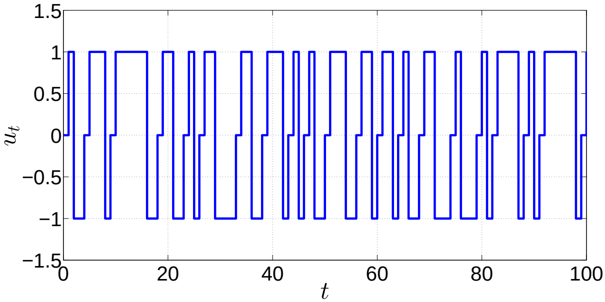
$$
\begin{aligned}
x_{t+1} &= \theta_1 x_t + \frac{x_t}{\theta_2 + x_t^2} + u_t + v_t, && v_t \sim \mathcal{N}(0, 0.1^2) \\
y_t &= \frac{1}{2} x_t + \frac{2}{5} x_t^2 + e_t, && e_t \sim \mathcal{N}(0, 0.1^2)
\end{aligned}
$$

where $\theta = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^\top$, $\theta_0 = \begin{bmatrix} 0.7 & 0.6 \end{bmatrix}^\top$.

Input design: $n_{\text{seq}} = 5 \cdot 10^3$, $n_m = 2$, $\mathcal{C} = \{-1, 0, 1\}$, and $h(\cdot) = \log\det(\cdot)$.

Example II

Input sequence:



Example II

Results:

Input / $h(\widehat{\mathcal{I}}_F)$	$\log\det(\widehat{\mathcal{I}}_F)$
Optimal	$\mathbf{25.34}$
Binary	24.75
Uniform	24.38

Problem formulation for output-error models

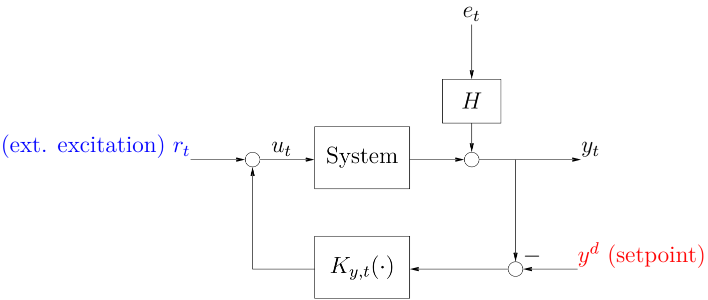
Input design based on graph theory

Extension to nonlinear SSM

Closed-loop application oriented input design

Conclusions and future work

Closed-loop application oriented input design



System ($\theta_0 \in \Theta$):

$$
\begin{aligned}
x_{t+1} &= A_{\theta_0} x_t + B_{\theta_0} u_t \\
y_t &= C_{\theta_0} x_t + \nu_t \\
\nu_t &= H(q;\, \theta_0) e_t
\end{aligned}
$$

$\{e_t\}$: white noise, known distribution.

Feedback: $u_t = r_t + K_{y,t}(y_t - y^d)$

Closed-loop application oriented input design

Model:

$$\begin{aligned} x_{t+1} &= A(\theta)x_t + B(\theta)u_t \\ y_t &= C(\theta)x_t + \nu_t \\ \nu_t &= H(q;\,\theta)e_t \end{aligned}$$

$\theta \in \Theta$.

Goal: Perform an experiment to obtain $\hat{\theta}_{n_{\mathrm{seq}}}$.
$\Rightarrow$ design $r_{1:n_{\mathrm{seq}}}$!

Requirements:

1. y_t, u_t should not be perturbed excessively.
2. $\hat{\theta}_{n_{\mathrm{seq}}}$ must satisfy quality constraints.

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Closed-loop application oriented input design

Minimize control objective:

$$J = \mathbf{E}\left\{\sum_{t=1}^{n_{\text{seq}}} \left\|y_t - y^d\right\|_Q^2 + \|u_t - u_{t-1}\|_R^2\right\}$$

Requirements:

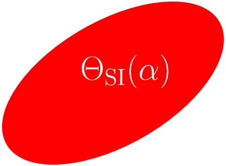
1. y_t, u_t should not be perturbed excessively.

Probabilistic bounds:

$$\mathbf{P}\{|y_t - y^d| \leq y_{\max}\} > 1 - \epsilon_y$$
$$\mathbf{P}\{|u_t| \leq u_{\max}\} > 1 - \epsilon_x$$

for $t = 1, \ldots, n_{\text{seq}}$

Closed-loop application oriented input design



Requirements:

2. $\hat{\theta}_{n_{\mathrm{seq}}}$ must satisfy quality constraints.

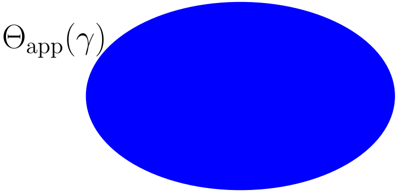
Maximum likelihood: As $n_{\mathrm{seq}} \to \infty$, we have:

$$\sqrt{n_{\mathrm{seq}}}\left(\hat{\theta}_{n_{\mathrm{seq}}} - \theta_0\right) \in \mathit{As}\mathcal{N}(0, \{\mathcal{I}_F^e\}^{-1})$$

$\Rightarrow$ Identification set:

$$\Theta_{\mathrm{SI}}(\alpha) = \left\{\theta : (\theta - \theta_0)^\top \mathcal{I}_F^e \, (\theta - \theta_0) \leq \chi^2_\alpha(n_\theta)\right\}$$

Closed-loop application oriented input design



Requirements:

2. $\hat{\theta}_{n_{\text{seq}}}$ must satisfy quality constraints.

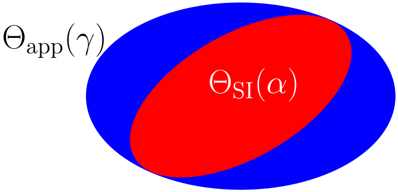
Quality constraint: Application set

$$\Theta(\gamma) = \left\{ \theta : V_{\text{app}}(\theta) \leq \frac{1}{\gamma} \right\}$$

Relaxation: Application ellipsoid

$$\Theta_{\text{app}}(\gamma) := \left\{ \theta : (\theta - \theta_0)^\top \, \nabla_\theta^2 V_{\text{app}}(\theta)\Big|_{\theta=\theta_0} (\theta - \theta_0) \leq \frac{2}{\gamma} \right\}$$

Closed-loop application oriented input design



Requirements:

2. $\hat{\theta}_{n_{\text{seq}}}$ must satisfy quality constraints.

Quality constraint:

$$\Theta_{\text{SI}}(\alpha) \subseteq \Theta_{\text{app}}(\gamma)$$

achieved by

$$\frac{1}{\chi^2_\alpha(n_\theta)} \mathcal{I}_F^e \succeq \frac{\gamma}{2} \left. \nabla^2_\theta V_{\text{app}}(\theta) \right|_{\theta=\theta_0}$$

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Closed-loop application oriented input design

Optimization problem:

$$\begin{aligned}
\min_{\{r_t\}_{t=1}^{n_{\text{seq}}}} \quad & J = \mathbf{E}\left\{\sum_{t=1}^{n_{\text{seq}}} \|y_t - y_d\|_Q^2 + \|\Delta u_t\|_R^2\right\} \\
\text{s.t.} \quad & \text{System constraints} \\
& \mathbf{P}\{|y_t - y^d| \leq y_{\max}\} > 1 - \epsilon_y \\
& \mathbf{P}\{|u_t| \leq u_{\max}\} > 1 - \epsilon_x \\
& \mathcal{I}_F \succeq \frac{\gamma \chi_\alpha^2(n_\theta)}{2} \nabla_\theta^2 V_{\text{app}}(\theta)
\end{aligned}$$

- Difficulty: $\mathbf{P}$ (and $\mathcal{I}_F^e$) hard to optimize.
- Solution: Use the graph-theory approach!

Closed-loop application oriented input design

Graph theory approach:

$r_{1:n_{\text{seq}}}$ realization from $p(r_{1:n_m})$ with alphabet $\mathcal{C}$.

To design an experiment in $\mathcal{C}^{n_m}$:

1. Compute all the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$.
2. Generate the signals $\{r_t^i\}_{t=0}^{t=N}$ from the prime cycles of $\mathcal{G}_{\mathcal{C}^{n_m}}$, for each $i \in \{1, \ldots, n_{\mathcal{V}}\}$.

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Closed-loop application oriented input design

Graph theory approach:

$r_{1:n_{\text{seq}}}$ realization from $p(r_{1:n_m})$ with alphabet $\mathcal{C}$.

Given $e_{1:N_{\text{sim}}}$, $r^{(j)}_{1:N_{\text{sim}}}$, approximate

$$J^{(j)} \approx \frac{1}{N_{\text{sim}}} \sum_{t=1}^{N_{\text{sim}}} \left\| y_t^{(j)} - y^d \right\|_Q^2 + \left\| u_t^{(j)} - u_{t-1}^{(j)} \right\|_R^2$$

$$\mathbf{P}_{e_t, r_t^{(j)}} \{ |u_t^{(j)}| \leq u_{\max} \} \approx \text{Monte Carlo}$$

$$\mathbf{P}_{e_t, r_t^{(j)}} \{ |y_t^{(j)} - y^d| \leq y_{\max} \} \approx \text{Monte Carlo}$$

$\mathcal{I}_F^{(j)}$ computed as in previous parts.

Closed-loop application oriented input design

Optimization problem (graph-theory):

$$
\begin{aligned}
\min_{\{\beta_1, \dots, \beta_{n_v}\}} \quad & \sum_{j=1}^{n_v} \beta_j J^{(j)} \\
\text{s.t.} \quad & \text{System constraints} \\
& \text{Constraints on } \{\beta_j\}_{j=1}^{n_v} \\
& \sum_{j=1}^{n_v} \beta_j \mathbf{P}_{e_t, r_t^{(j)}} \{|u_t^{(j)}| \leq u_{\max}\} > 1 - \epsilon_x \\
& \sum_{j=1}^{n_v} \beta_j \mathbf{P}_{e_t, r_t^{(j)}} \{|y_t^{(j)} - y^d| \leq y_{\max}\} > 1 - \epsilon_y \\
& \sum_{j=1}^{n_v} \beta_j \mathcal{I}_F^{(j)} \succeq \frac{\gamma \chi_\alpha^2(n)}{2 n_{\text{seq}}} \nabla_\theta^2 V_{\text{app}}(\theta)
\end{aligned}
$$

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Closed-loop application oriented input design

Optimal pmf:

$$p^{\text{opt}} := \sum_{j=1}^{n_v} \beta_j^{\text{opt}} p_j$$

Example III

Consider the open-loop, SISO state space system

$$x_{t+1} = \theta_2^0 \, x_t + u_t$$
$$y_t = \theta_1^0 \, x_t + e_t$$

$\begin{bmatrix} \theta_1^0 & \theta_2^0 \end{bmatrix}^\top = \begin{bmatrix} 0.6 & 0.9 \end{bmatrix}^\top.$

Input:

$$u_t = r_t - k_y \, y_t$$

$k_y = 0.5$ known.

Goal: Estimate $\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^\top$ using indirect identification.

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Example III

Design $\{r_t\}_{t=1}^{500}$, $n_m = 2$, $r_t \in \mathcal{C} = \{-0.5, -0.25, 0, 0.25, 0.5\}$.

Performance degradation:

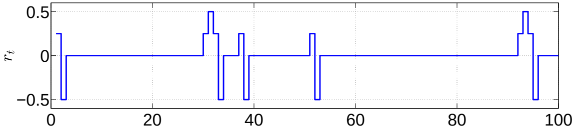
$$V_{\text{app}}(\theta) = \frac{1}{500} \sum_{t=1}^{500} \|y_t(\theta_0) - y_t(\theta)\|_2^2$$

- $y^d = 0$
- $\epsilon_y = \epsilon_x = 0.07$
- $y_{\max} = 2$, $u_{\max} = 1$

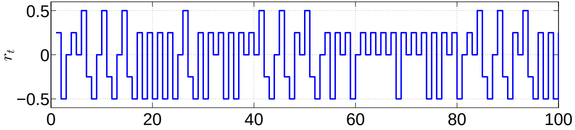
Example III

Reference:

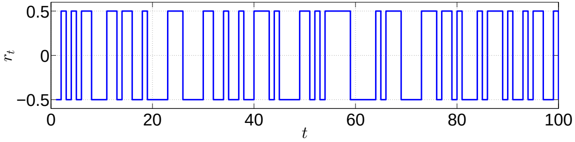
Opt. 1:



Opt. 2:



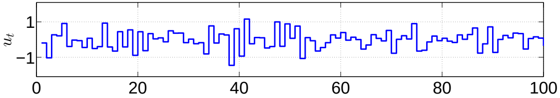
Binary:



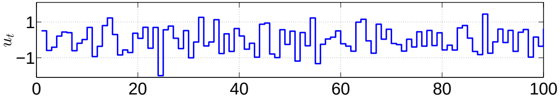
Example III

Input:

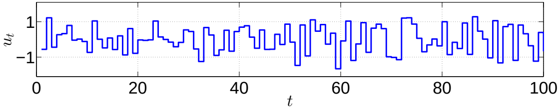
Opt. 1: 93.8%



Opt. 2: 86.6%



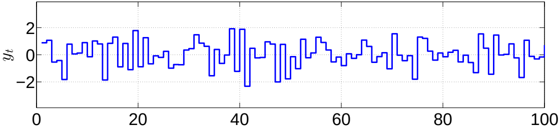
Binary: 90.8%



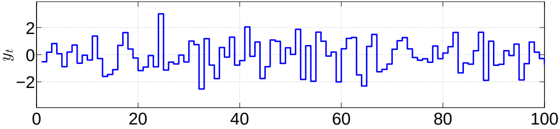
Example III

Output:

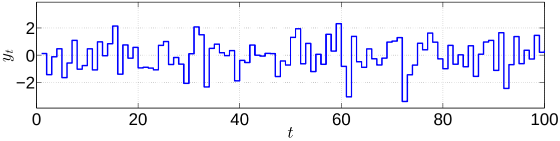
Opt. 1: 96%



Opt. 2: 93.4%

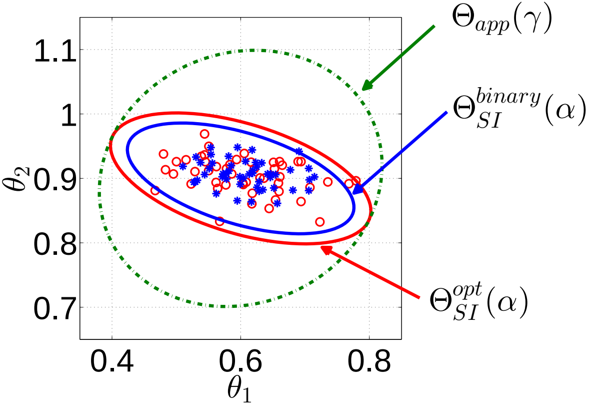


Binary: 79.6%



Example III

Application and identification ellipsoids: (98% confidence level)



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Conclusions

- A new method for input design was introduced.
- The method can be used for nonlinear systems.
- Convex problem even for nonlinear systems.

Future work

- Reducible Markov chains.
- Computational complexity.
- Robust input design.
- Application oriented input design for MPC.

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Thanks for your attention.

Optimal input design for nonlinear dynamical systems: a graph-theory approach

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Seminar Uppsala university
January 16, 2015

Outline

Problem formulation for output-error models

Input design based on graph theory

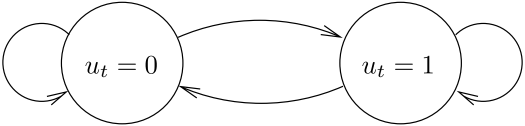
Extension to nonlinear SSM

Closed-loop application oriented input design

Conclusions and future work

Appendix I: Graph theory in input design

Example: Elementary cycles for a de Bruijn graph, $\mathcal{C} := \{0, 1\}$, $n_m := 1$.



One elementary cycle: $z = (0, 1, 0)$.

$\Rightarrow$ Prime cycles for a de Bruijn Graph, $\mathcal{C} := \{0, 1\}$, $n_m := 2$: $v_1 = ((0, 1), (1, 0), (0, 1))$.

Appendix II: Graph theory in input design

Example: Generation of input signal from a prime cycle.

Consider a de Bruijn graph, $\mathcal{C} := \{0, 1\}$, $n_m := 2$.

- $v_1 = ((0, 1), (1, 0), (0, 1))$
- $\{u_t^1\}_{t=0}^{t=N}$: Take last element of each node.

Finally,

$$\{u_t^i\}_{t=0}^{t=N} = \{1, 0, 1, 0, \ldots, ((-1)^N + 1)/2\}$$

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Appendix III: Building A

- For $i \in \mathcal{C}^{n_m}$, define

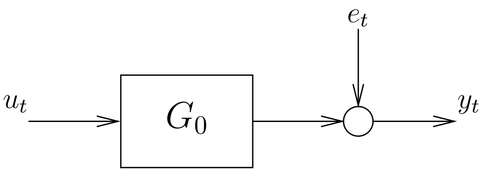
$$\mathcal{A}_i := \{j \in \mathcal{C}^{n_m} : (j, i) \in \mathcal{E}\}.$$

(the set of ancestors of i).

- For each $i \in \mathcal{C}^{n_m}$, let

$$A_{ij} = \begin{cases} \dfrac{\mathbf{P}\{i\}}{\sum\limits_{k \in \mathcal{A}_i} \mathbf{P}\{k\}}, & \text{if } j \in \mathcal{A}_i \text{ and } \sum\limits_{k \in \mathcal{A}_i} \mathbf{P}\{k\} \neq 0 \\ \dfrac{1}{\# \mathcal{A}_i}, & \text{if } j \in \mathcal{A}_i \text{ and } \sum\limits_{k \in \mathcal{A}_i} \mathbf{P}\{k\} = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Apendix IV: Example nonlinear case



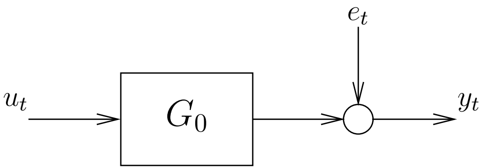
$$G_0(u_t) = G_1(q,\theta)\, u_t + G_2(q,\theta)\, u_t^2$$

where

$$G_1(q,\theta) = \theta_1 + \theta_2\, q^{-1}$$
$$G_2(q,\theta) = \theta_3 + \theta_4\, q^{-1}$$

e_t: Gaussian white noise, zero mean, variance $\lambda_e = 1$.

Apendix IV: Example nonlinear case



$$G_0(u_t) = G_1(q,\theta)\, u_t + G_2(q,\theta)\, u_t^2$$

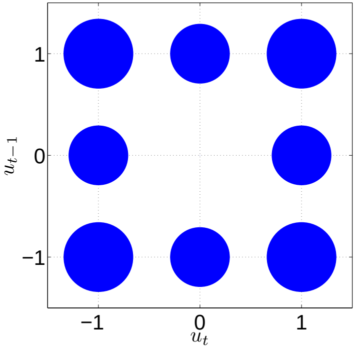
where

$$G_1(q,\theta) = \theta_1 + \theta_2\, q^{-1}$$
$$G_2(q,\theta) = \theta_3 + \theta_4\, q^{-1}$$

We consider $h(\cdot) = \det(\cdot)$, $c_{\text{seq}} = 3$, $n_m = 2$, $\mathcal{C} := \{-1, 0, 1\}$, and $N = 5 \cdot 10^3$.

Apendix IV: Example nonlinear case

Stationary probabilities:



- $\det(\mathcal{I}_F^{\text{app}}) = 0.1796.$
- Results consistent with previous contributions (Larsson et al. 2010).

Appendix V: Particle methods to estimate $\mathcal{S}_m(\theta_0)$

Auxiliary particle filter:

$$\hat{p}_\theta(x_{1:t}|y_{1:t}) := \sum_{i=1}^{N} \frac{w_t^{(i)}}{\sum_{k=1}^{N} w_t^{(k)}} \delta(x_{1:t} - x_{1:t}^{(i)})$$

$\{x_{1:t}^{(i)}, w_t^{(i)}\}_{i=1}^N$: Particle system.

Two step procedure to compute $\{x_{1:t}^{(i)}, w_t^{(i)}\}_{i=1}^N$:

1. Sampling/propagation.

2. Weighting.

Appendix V: Particle methods to estimate $\mathcal{S}_m(\theta_0)$

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Two step procedure to compute $\{x_{1:t}^{(i)},\, w_t^{(i)}\}_{i=1}^N$:

1. Sampling/propagation:

$$\{a_t^{(i)},\, x_t^{(i)}\} \sim \frac{w_{t-1}^{a_t}}{\sum_{k=1}^N w_{t-1}^{(k)}} R_{\theta,t}(x_t | x_{t-1}^{a_t}, u_{t-1})$$

Appendix V: Particle methods to estimate $\mathcal{S}_m(\theta_0)$

Two step procedure to compute $\{x_{1:t}^{(i)},\, w_t^{(i)}\}_{i=1}^N$:

2. Weighting:

$$w_t^{(i)} := \frac{g_\theta(y_t|x_t^{(i)}, u_t)\, f_\theta(x_t|x_{t-1}^{(i)}, u_{t-1})}{R_{\theta,t}(x_t|x_{t-1}^{(i)}, u_{t-1})}$$

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Appendix V: Particle methods to estimate $\mathcal{S}_m(\theta_0)$

- Difficulty: Auxiliary particle filter suffers particle degeneracy.

- Solution: Use fixed-lag smoother.

Main idea FL-smoother:

$$p_\theta(x_t|y_{1:T}, u_{1:T}) \approx p_\theta(x_t|y_{1:\kappa_t}, u_{1:\kappa_t})$$

for $\kappa_t = \min(t + \Delta, T)$, for some fixed-lag $\Delta > 0$.

Apendix VI: Equivalence time and frequency domain

	Frequency	Time
Design variable	$\Phi_u(\omega)$	$p(u_{1:n_m})$
Restrictions	$\Phi_u(\omega) \geq 0$ $\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq 1$	$p \geq 0$ $\sum_{u_{1:n_m}} p(u_{1:n_m}) = 1$ p stationary
Information matrix	$\mathcal{I}_F(\Phi_u)$	$\mathcal{I}_F(p)$