Applications Oriented Input Design for Closed-Loop System Identification: a Graph-Theory Approach

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***Abstract*— A new approach to experiment design for identification of closed-loop models is presented. The method considers the design of an experiment by minimizing experimental cost, subject to probabilistic bounds on the input and output signals, and quality constraints on the identified model. The input and output bounds are common in many industrial processes due to physical limitations of actuators. The aforementioned constraints make the problem non-convex. By assuming that the experiment is a realization of a stationary process with finite memory and finite alphabet, we use results from graph-theory to relax the problem. The key feature of this approach is that the problem becomes convex even for non-linear feedback systems. A numerical example shows that the proposed technique is an attractive alternative for closed-loop system identification.**

I. Introduction

System identification concerns the problem of plant modelling based on collected data, with a broad application in industry [1]. The collected data for identification could be gathered under either open- or closed-loop operation. The later case has been of prime interest in many industrial applications.

In practical applications, many systems can only work on closed-loop settings due to stability issues, production restrictions, economic considerations or inherent feedback mechanisms. On the other hand, it is sometimes required to update the existing control laws or design a new controller. Since most of the methods for analysing controllers require the knowledge of the system to be controlled, closed-loop system identification is a building block in this process. The main burden in closed-loop identification is the correlation between the measurement noise and input signal, which is imposed on the experiment by the feedback loop. There is a quite rich literature on closed-loop identification with three main approaches: direct methods (the model is identified as if the system were in open-loop), indirect methods (the model is identified from the identified closed-loop structure), and joint input-output (an augmented model is identified, where the input and output of the system are considered as the new outputs, and the reference and noise as new inputs); see e.g. [1], [2], [3], [4] and the references therein.

One crucial question that arises in any identification process is how to generate data efficiently. This question is addressed by input design methods, where the objective is to generate an input signal that maximizes the information retrieved from an experiment [5], [6]. In this area, applications oriented input design is one approach to formulate the optimal input design problem. The main idea is to guarantee a certain control performance for the identified model with the least possible experimental effort. The same idea has been used in identification for control and least costly identification, see e.g. [6], [7], [8].

For closed-loop models, the input design problem is often translated to design the spectrum of an additive external excitation. There exists a rich literature on closed-loop experiment design, where the controller can also be designed (provided it is a design variable), see [9], [10], [11], [12] and references therein. However, the main limitation on the existing methods is that they cannot be employed in closed-loop systems with nonlinear feedback. In addition, they cannot handle probabilistic constraints on the input and output, which arise for safety or practical limitations.

In this article we present a new approach for applications oriented experiment design for closed-loop systems. We consider a linear time-invariant model being controlled by a known controller (either linear or non-linear), where the main goal is reference tracking. Due to a performance degradation, we want to update the current controller or design another one, and thus the plant model needs to be identified. Since the controller is known we will employ indirect identification, where the model is identified by adding an external stationary input. The problem is then formulated as an optimization, where we design the external excitation achieving the minimum experimental effort, while we are also taking care of the tracking performance of the existing controller. We add a constraint on the quality of the estimated model in terms of the Fisher information matrix [1], to get an exciting enough input signal guaranteeing that the estimated model is in the set of models that satisfies the desired control specifications, with a given probability.

In practice we also have bounds on the input and output signals, which should be taken into account during the experiment design. Thus, the optimization also considers probabilistic bounds for the input and output of the system.

The obtained optimization problem is non-convex due to the constraints, and thus it is difficult to handle. This issue is relaxed by extending the method introduced in [13] for closed-loop and constrained system identification, where the probability distribution function associated with the external excitation is characterized as the convex combination of the measures describing the set of stationary processes. The resulting problem is convex on the decision variables, which makes it tractable. The method allows us to use Monte-Carlo

This work was supported in part by the Swedish Research Council under contracts 621-2011-5890 and 621-2009-4017, by the European Research Council under the advanced grant LEARN, contract 267381, and by the European Unions Seventh Framework Programme (FP7/2007-2013) under grant agreement no 257059, the Autoprofit project (www.fp7-autoprofit.eu).

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methods to approximate the cost functions, probabilities and information matrices associated with each extreme measure.

The paper is organized as follows. In Section II we introduce the concepts employed in this work. Section III presents the proposed method for closed-loop experiment design. Section IV illustrates the results through a numerical example. Finally, Section V presents conclusions and future work on the subject.

II. Preliminaries

Consider the discrete time, linear, time-invariant system

$$\begin{aligned} x_{t+1} &= A(\theta_o)x_t + B(\theta_o)u_t, \\ y_t &= C(\theta_o)x_t + v_t. \end{aligned} \tag{1}$$

where $u_t \in \mathbb{R}^{n_u}$ and $y_t \in \mathbb{R}^{n_y}$ are the input and output vectors. $v_t \in \mathbb{R}^{n_e}$ is a coloured noise with $v_t = H(q;\theta_o)e_t$, where H is a rational noise filter in terms of the time shift operator q, and $\{e_t\}$ is white noise sequence with zero mean and covariance matrix Λ_e. In addition, we assume that H is stable, inversely stable, and satisfies $H(\infty;\theta) = I$.

A. System Identification

In system identification, we aim to find a model of the system (1). The model is parameterized by an unknown parameter vector $\theta \in \mathbb{R}^{n_\theta}$, that is,

$$\begin{aligned} x_{t+1} &= A(\theta)x_t + B(\theta)u_t\,, \\ y_t &= C(\theta)x_t + v_t\ , \end{aligned} \tag{2}$$

where $v_t = H(q;\theta)e_t$. The model coincides with (1) exactly when $\theta = \theta_o$ [1].

We employ the prediction error method (PEM) with quadratic cost to calculate an approximation of the unknown parameters $\theta \in \mathbb{R}^{n_\theta}$, based on N available samples of input-output, i.e. the data $\{u_t, y_t, t = 1,\dots,N\}$ [1]. An important asymptotic (in the sample size N) property of PEM, is that

$$\sqrt{N}(\theta - \theta_o) \sim AsN(0, \mathfrak{I}_F^{-1}(\theta_o)), \tag{3}$$

where $\mathfrak{I}_F$ quantify the information regarding the unknown parameters, θ, in the observations of the output signal which is called Fisher information matrix. Thus, for sufficiently large samples N, we get that with a certain probability α the estimated parameters belongs to an *identification set*, (see [14]), defined as

$$\mathcal{E}_{SI}(\alpha) = \left\{\theta : [\theta - \theta_o]^T \mathfrak{I}_F(\theta_o)[\theta - \theta_o] \le \chi^2_\alpha(n_\theta)\right\}, \tag{4}$$

where $\chi^2_\alpha(n)$ is the α-percentile of the χ^2-distribution with n degrees of freedom, which in turn implies that $\hat{\theta}_N \in \mathcal{E}_{SI}(\alpha)$ with probability α for sufficiently large samples. For more details, we refer the reader to [1].

B. Applications Oriented Input Design

In applications oriented input design, the main focus is to design an input signal to be used in identification experiment such that an acceptable control performance can be guaranteed when the estimated model is used in the control design. This requires that $\hat{\theta}_N \in \Theta(\gamma)$ with high probability, where $\Theta(\gamma)$, also known as *application set*, is the set of all acceptable parameters from control's point of view, and γ is a user-defined positive constant which imposes an upper bound on the performance degradation. One way to ensure this is to require

$$\mathcal{E}_{SI}(\alpha) \subseteq \Theta(\gamma). \tag{5}$$

Using (5), the input design problem can be formulated as a constrained optimization problem with (5) as the constraint. Thus, a natural objective in the input design is to minimize an experimental cost, such as input power or energy or experimental time, while (5) is fulfilled, i.e.

$$\begin{aligned} \min_{\text{input}} \quad & \text{Experimental Cost} \\ \text{s.t.} \quad & \mathcal{E}_{SI}(\alpha) \subseteq \Theta(\gamma). \end{aligned} \tag{6}$$

In order to relate the control performance degradation to the plant-model mismatch, we use the concept of application cost function, where a scalar function of θ is considered as application cost, denoted by $V_{app}(\theta)$. We choose the cost function such that its minimum value occurs at $\theta = \theta_o$. In particular, if $V_{app}(\theta)$ is twice differentiable in a neighbourhood of θ_o, we assume without loss of generality:

$$V_{app}(\theta_o) = 0, V'_{app}(\theta_o) = 0 \ \text{ and } \ V''_{app}(\theta_o) \ge 0.$$

There are many possible choices of application functions with these properties, see e.g. [15]. The set of all acceptable parameters, namely the *application set*, is defined as

$$\Theta(\gamma) = \left\{\theta : V_{app}(\theta) \le \frac{1}{\gamma}\right\}. \tag{7}$$

To proceed, we employ the following local approximation of $\Theta(\gamma)$ invoking the Taylor expansion of $V_{app}(\theta)$ around θ_o:

$$\begin{aligned} V_{app}(\theta) &\approx V_{app}(\theta_o) + V'_{app}(\theta_o)[\theta - \theta_o] \\ &\quad + 0.5[\theta - \theta_o]^T V''_{app}(\theta_o)[\theta - \theta_o] \\ &= 0 + 0 + 0.5[\theta - \theta_o]^T V''_{app}(\theta_o)[\theta - \theta_o]. \end{aligned} \tag{8}$$

Thus we have the following ellipsoidal approximation of the application set (see [7]):

$$\Theta(\gamma) \approx \mathcal{E}_{app}(\gamma) = \left\{\theta : [\theta - \theta_o]^T V''_{app}(\theta_o)[\theta - \theta_o] \le \frac{2}{\gamma}\right\}, \tag{9}$$

and therefore, the optimal input design problem (6) can be rewritten as

$$\begin{aligned} \min_{\text{input}} \quad & \text{Experimental Cost} \\ \text{s.t.} \quad & \frac{1}{\chi^2_\alpha(n_\theta)}\mathfrak{I}_F(\theta_o) \ge \frac{\gamma}{2}V''_{app}(\theta_o). \end{aligned} \tag{10}$$

For more details on applications oriented input design we refer the reader to [7].

C. Optimal Input Design via Graph Theory

In input design via graph theory the optimal input signal $\mathcal{R}_M = (r_M, \ \dots, \ r_1)$ is designed as a realization of a stationary process with finite memory [13]. The problem is then formulated in terms of the projected probability distribution function (pdf) of the stationary input, denoted by $P(\mathcal{R}_{n_m})$, where $n_m < M$. Assuming that the input signal belongs to a finite set of values (say $\mathcal{C}$), we can use elements from graph theory to describe any $P(\mathcal{R}_{n_m})$ in the set of

stationary processes as a convex combination of the extreme points of the set. The extreme points are computed as the prime cycles associated with the de Brujin Graph of memory n_m and alphabet $\mathcal{C}$ (see [13]). The main advantage of this approach is that it results in a convex optimization problem for convex objective functions even for nonlinear models.

III. Applications Oriented Input design for Closed-Loop System Identification

A. Problem Definition

Assume that the system (2) is controlled using a general (either linear or non-linear) output feedback controller:

$$u_t = r_t - K_y(y^t), \tag{11}$$

where K_y is a known function, and $y^t := \{y_k\}_{k=1}^t$. The feedback (11) is such that the output signal tracks a desired value y_d. The closed-loop structure is shown in Figure 1.

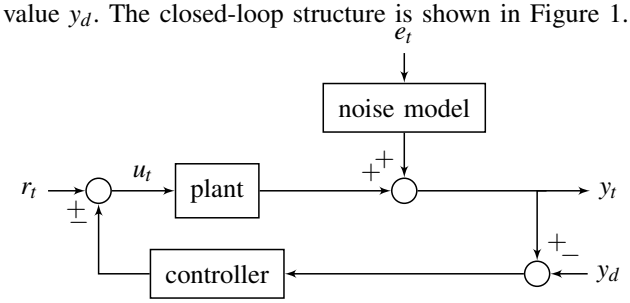

Fig. 1. The schematic representation of the closed-loop system

Thus the closed-loop system will be

$$\begin{aligned} x_{t+1} &= F(\theta, x_t, y^t) + B(\theta) r_t, \\ y_t &= C(\theta) x_t + \nu_t, \end{aligned} \tag{12}$$

where $\nu_t = H(q;\theta)e_t$, and

$$F(\theta, x_t, y^t) := A(\theta)x_t - B(\theta)K_y(y^t).$$

We assume that the resulting closed-loop system (12) is asymptotically stable.

The objective in this article is to design an experiment for the closed-loop system (12), that generates M samples of the reference signal r_t, to be used for identification of the unknown parameters θ in (2). To this end, we consider the experiment design problem (10). Since the system is in the closed-loop we need to keep the output of the plant y_t close to y_d during the identification experiment. Hence, we choose to minimize the following experimental cost in the optimal input design problem (10)

$$J = \mathbb{E}\left\{ \sum_{t=1}^{M} \|y_t - y_d\|_Q^2 + \|\Delta u_t\|_R^2 \right\}, \tag{13}$$

where

$$\Delta u_t = u(t) - u(t-1), \tag{14}$$

and Q and R are positive definite matrices. The first term in (13) penalizes the deviations from the desired output, while the second term is responsible for minimizing the input energy. The expected value is with respect to $\{r_t\}$ and $\{e_t\}$.

In practical applications, it is common to have bounds on the maximal input and output amplitudes allowed by the process. These constraints appear due to physical limitations and/or to preserve the system in a safe operating point. Thus we consider the following probabilistic constraints during the identification process:

$$\begin{aligned} &\mathbb{P}\{|y_t - y_d| \le y_{max}\} > 1 - \varepsilon_y,\ t = 1,\dots,M, \\ &\mathbb{P}\{|u_t| \le u_{max}\} > 1 - \varepsilon_x,\ t = 1,\dots,M, \end{aligned} \tag{15}$$

where u_{max} is the maximum allowed value for the input signal and y_{max} is the maximum allowed deviation of the output from its desired value, based on the physical properties of the system and actuators; and ε_x with ε_y are two design variables that define the desired probability of being in the safe bounds for input and output signals.

In addition to the previous constraints, we require that the updated (or new) designed controller based on the estimated parameters can guarantee an acceptable control performance, i.e. the experiment design constraint (5) is satisfied. The optimization problem can be summarized as:

Problem 1: Design $\{r_t^{\text{opt}}\}_{t=1}^M$ as the solution of

$$\begin{aligned} \min_{\{r_t\}_{t=1}^M} \quad & J = \mathbb{E}\left\{ \sum_{t=1}^{M} \|y_t - y_d\|_Q^2 + \|\Delta u_t\|_R^2 \right\}, \\ \text{s.t.} \quad & x_{t+1} = F(\theta, x_t, y^t) + B(\theta) r_t, \\ & y_t = C(\theta) x_t + \nu_t,\ t = 1,\dots,M, \\ & \nu_t = H(q;\theta)e_t,\ t = 1,\dots,M, \\ & u_t = r_t - K_y(y^t),\quad t = 1,\dots,M, \\ & \mathbb{P}\{|y_t - y_d| \le y_{max}\} > 1 - \varepsilon_y,\ t = 1,\dots,M, \\ & \mathbb{P}\{|u_t| \le u_{max}\} > 1 - \varepsilon_x,\ t = 0,\dots,M-1, \\ & \mathfrak{I}_F(\theta) \ge \frac{\gamma \chi_\alpha^2(n)}{2} V_{app}^{"}(\theta), \end{aligned} \tag{16}$$

where $\mathfrak{I}_F(\theta)$ is the Fisher information matrix obtained with M samples. ■

Note that Problem 1, has a very similar structure as Model Predictive Control, see [16]. However, they are not necessarily the same since we are not considering a receding horizon approach in this problem.

The optimization problem (16) is non-convex due to the possible non-linearity of the closed-loop system and the experiment design constraints and is difficult to be solved explicitly.

Remark 1: Problem 1 relies on the knowledge of the true system. This can be addressed by either implementing a robust experiment design scheme on top of it [17]; or through an adaptive procedure, where the Hessian of the cost function and output predictions are updated as more information is being collected, [18]. In the rest of this paper we rely on the knowledge of θ_o (or a prior estimate of it). ■

B. Convex Relaxation of the Optimization Algorithm

To find a convex relaxation of Problem 1 we will use elements from graph theory [13] to design an input sequence $(r_1, \dots, r_M)$ such that the cost function in (16) is minimized, satisfying input-output requirements, and identification constraints. To proceed we will assume that $(r_1, \dots, r_M)$ is

a realization from a stationary pdf $P\{r_1,\dots,r_{n_m}\}$, where $n_m < M$ is the memory of the stationary process. In addition, we assume that $r_t \in \mathcal{C}$, for $t \in \{1,\dots,M\}$, with $\mathcal{C}$ defined as a finite set. Under the previous assumptions, we can use the theory introduced in [13] to define the set of feasible pdf's $P\{r_1,\dots,r_{n_m}\}$ as a convex combination of the extreme points in the set. The extreme points are computed as the prime cycles associated with the de Brujin graph of memory n_m and alphabet $\mathcal{C}$ [13].

If the set of extreme points is given by $\{P_j\}_{j=1}^{n_v}$, then the optimization problem can be rewritten as

$$\begin{aligned}
\min_{\{\beta_1,\dots,\beta_{n_v}\}} \quad & J = \sum_{t=1}^{M}\sum_{j=1}^{n_v} \beta_j \mathbb{E}_{e_t,P_j}\left\{ \left\| y_t - y^d \right\|_Q^2 + \|\Delta u_t\|_R^2 \right\}, \\
\text{s.t.} \quad & x_{t+1} = F(\theta, x_t, e_t) + B(\theta) r_t, \\
& y_t = C(\theta) x_t + v_t,\ t = 1,\dots,M, \\
& v_t = H(q;\theta) e_t,\ t = 1,\dots,M, \\
& u_t = r_t - K_y(y^t),\quad t = 1,\dots,M, \\
& \sum_{j=1}^{n_v} \beta_j \mathbb{P}_{e_t,P_j}\{|u_t| \le u_{max}\} > 1-\varepsilon_x, \\
& \sum_{j=1}^{n_v} \beta_j \mathbb{P}_{e_t,P_j}\{|y_t - y_d| \le y_{max}\} > 1-\varepsilon_y, \\
& \sum_{j=1}^{n_v} \beta_j \mathfrak{I}_F^{(j)}(\theta) \ge \frac{\gamma \chi_\alpha^2(n)}{2} V_{app}^{"}(\theta), \\
& \sum_{j=1}^{n_v} \beta_j = 1, \beta_j \ge 0,\ j = 1,\dots,n_v.
\end{aligned} \tag{17}$$

If we denote by $\{\beta_j^{\text{opt}}\}_{j=1}^{n_v}$ the set of weighting factors minimizing (17), then the optimal pdf is given by

$$P^{\text{opt}} := \sum_{j=1}^{n_v} \beta_j^{\text{opt}} P_j . \tag{18}$$

Since the set $\{P_j\}_{j=1}^{n_v}$ is known (each P_j is a uniform distribution over the nodes in the j-th prime cycle [13]), we can sample $\{r_t^j\}_{t=1}^N$ from each P_j (with N sufficiently large), and approximate by Monte-Carlo methods the expected values $\mathbb{E}_{P_j}\{\cdot\}$, the probabilities $\mathbb{P}_{P_j}\{\cdot\}$, and the corresponding information matrices $\mathfrak{I}_F^{(j)}(\theta)$. This approach is based on the one presented in [13], where Monte-Carlo simulations are employed to compute the information matrices associated with each measure in the set $\{P_j\}_{j=1}^{n_v}$. Indeed, given $\{r_t^j\}_{t=1}^N$ and $\{e_t\}_{t=1}^N$, we can generate $\{y_t^j\}_{t=1}^N$ and $\{u_t^j\}_{t=1}^N$ using (12). Therefore, we can use Monte-Carlo approximations to compute the expressions in (17) for each $j \in \{1,\dots,n_v\}$ as

$$\begin{aligned}
\mathbb{P}_{P_j}\{|y_t - y_d| \le y_{max}\} &\approx \frac{1}{N}\sum_{t=1}^{N} \mathbf{1}_{|y_t^j| \le y_{max}}, \\
\mathbb{P}_{P_j}\{|u_t| \le u_{max}\} &\approx \frac{1}{N}\sum_{t=1}^{N} \mathbf{1}_{|u_t^j| \le u_{max}}, \\
\mathbb{E}_{P_j}\left\{ \|y_t - y_d\|_Q^2 + \|\Delta u_t\|_R^2 \right\} &\approx \frac{1}{N}\sum_{t=1}^{N} \left\| y_t^j - y_d \right\|_Q^2 + \left\| \Delta u_t^j \right\|_R^2,
\end{aligned}$$

where $\mathbf{1}_X = 1$ if X is true, and 0 otherwise, and $\Delta u_t^j = u_t^j - u_{t-1}^j$. The computation of $\mathfrak{I}_F^{(j)}(\theta)$ is analyzed in the next subsection.

The key property of the proposed approach is that the input-output constraints and the restriction on the Fisher information matrix are convex on $\{\beta_j\}_{j=1}^{n_v}$. Therefore, the final optimization problem becomes convex in $\{\beta_j\}_{j=1}^{n_v}$.

C. Fisher Information Matrix Computation

To integrate the experiment design constraint with the optimization problem (17), we need to compute the Fisher information matrix $\mathfrak{I}_F^{(j)}(\theta)$ for each $\{r_t^j\}_{t=1}^M$ associated with the j-th extreme point of the set of stationary processes of memory n_m and alphabet $\mathcal{C}$.

For an unbiased estimator, the inverse of the Fisher matrix is a lower bound on the covariance of the parameter estimation error, according to the Cramér-Rao bound. The Fisher information matrix is [5]

$$\mathfrak{I}_F(\theta) := \mathbb{E}\left\{ \frac{\partial \log p_\theta(y^M)}{\partial \theta} \frac{\partial \log p_\theta(y^M)}{\partial \theta^T} \right\} \in \mathbb{R}^{n_\theta \times n_\theta}, \tag{19}$$

where $y^M := \{y_1,\dots,y_M\}$. We notice that (19) can also be written as

$$\mathfrak{I}_F(\theta) = -\mathbb{E}\left\{ \frac{\partial^2 \log p_\theta(y^M)}{\partial \theta \partial \theta^T} \right\} \in \mathbb{R}^{n_\theta \times n_\theta}. \tag{20}$$

Due to the randomness of e_t, (12) can be rewritten as

$$\begin{aligned}
x_{t+1} &\sim f_\theta(x_{t+1}|y^t, x_t, r_t), \\
y_t &\sim g_\theta(y_t|x_t),
\end{aligned} \tag{21}$$

where $f_\theta(x_{t+1}|x_t, r_t)$ and $g_\theta(y_t|x_t)$ denotes the pdf of the state x_{t+1}, and output y_t, conditioned on the knowledge of $\{x_t, r_t\}$.

Using the model description (21), and the Markov property of the system (12), we can write the log likelihood as

$$\begin{aligned}
\log p_\theta(y^M) = &\sum_{t=1}^{M-1} \log\{f_\theta(x_{t+1}|y^t, x_t, r_t)\} \\
&+ \sum_{t=1}^{M} \log\{g_\theta(y_t|x_t)\} + \log\{p_\theta(x_1)\}.
\end{aligned} \tag{22}$$

To simplify the analysis, we will assume that the distribution of the initial state is independent of θ, i.e., $p_\theta(x_1) = p(x_1)$.

When K_y is a linear controller, expressions (19) and (20) can be computed in the frequency domain [1, Section 9.4]. Since we know $\{r_t^j\}$ for each $j \in \{1,\dots,n_v\}$, it is possible to compute its corresponding spectrum, say $\Phi_r^j(\omega)$. To this end, we notice that $\{r_t^j\}$ is a periodic sequence with period given by T_j. Using [1, Example 2.3] we compute $\Phi_r^j(\omega)$ as

$$\Phi_r^j(\omega) = \frac{2\pi}{T_j} \sum_{k=0}^{T_j-1} \Phi_r^{j,p}(2\pi k/T_j)\delta(\omega - 2\pi k/T_j),\ 0 \le \omega < 2\pi, \tag{23}$$

where δ is the Dirac delta function, and

$$\Phi_r^{j,p}(\omega) := \sum_{\tau=0}^{T_j-1} R_r^j(\tau) e^{i\omega\tau}, \tag{24}$$

$$R_r^j(\tau) := \frac{1}{T_j}\sum_{t=1}^{T_j} r_t^j \left(r_{t-\tau}^j \right)^T. \tag{25}$$

On the other hand, when K_y is a non-linear function, equations (19) and (20) often result in complex (and almost intractable) expressions. Thus, we will approximate the Fisher information matrix using numerical methods, instead. One solution is to use particle methods to approximate (19) as the covariance matrix of the gradient of the log-likelihood function, $\frac{\partial \log p_\theta(y^M)}{\partial \theta}$ (*score* function) [19]. Another approach is based on the numerical computation of (20) using small perturbation methods [20], where the Hessian (20) is computed as an average of numerical approximations based on the score function. Thus, the Fisher information matrix associated with $\{r_t^j\}$ can be computed for nonlinear systems.

IV. Numerical Example

To illustrate the previous discussion, we introduce the following example:

Example 1: Consider the open-loop, SISO state space system described by

$$x_{t+1} = \theta_2^0 x_t + u_t\,, \tag{26a}$$
$$y_t = \theta_1^0 x_t + e_t\,, \tag{26b}$$

with true parameters $\theta_0 := \begin{bmatrix}\theta_1^0 & \theta_2^0\end{bmatrix}^T = \begin{bmatrix}0.6 & 0.9\end{bmatrix}^T$. The system is controlled in closed-loop using the controller

$$u_t = r_t - k_y y_t\,, \tag{27}$$

where $k_y = 0.5$ is a known constant. The objective is to identify the open-loop parameters $\theta := \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^T$ from the identified closed-loop ones $\theta_c := \begin{bmatrix}\theta_1^c & \theta_2^c\end{bmatrix}^T$ in the model

$$x_{t+1} = \theta_2^c x_t + r_t - k_y e_t\,, \tag{28a}$$
$$y_t = \theta_1^c x_t + e_t\,, \tag{28b}$$

using the transformation law

$$\theta_1 = \theta_1^c\,, \tag{29a}$$
$$\theta_2 = \theta_1^c + k_y \theta_1^c\,. \tag{29b}$$

To this end, we will design the reference signal $\{r_t\}_{t=1}^{500}$ as a realization of a stationary process with memory $n_m = 2$, and subject to $r_t \in \mathcal{C} := \{-0.5, -0.25, 0, 0.25, 0.5\}$, for all $t \in \{1, \ldots, 500\}$. Since the experiment will be performed in closed-loop, we define the following cost function to measure performance degradation

$$V_{app}(\theta) := \frac{1}{500}\sum_{t=1}^{500} \|y_t(\theta_o) - y_t(\theta)\|_2^2\,, \tag{30}$$

where $y_t(\theta)$ denotes the closed-loop output when θ is employed to describe the open loop model and a linear output feedback controller with constant and θ-independent gain has been used. Finally, we will solve the approximate problem (17), where $y_d = 0$, for all $t \in \{1, \ldots, 500\}$, $Q = 1$, $R = 0.02$, $\varepsilon_y = \varepsilon_x = 0.07$, $y_{max} = 2$, $u_{max} = 1$, $\gamma = 10^2$, and $\alpha = 0.98$.

Figure 2 presents one realization of $\{r_t\}_{t=1}^{500}$ obtained by solving (17), one realization of $\{r_t\}_{t=1}^{500}$ obtained by solving (17) without probabilistic constraints, and from a random binary sequence with values $\{-0.5, 0.5\}$. From this figure we see that the optimal sequence is zero most of the time, except for short pulses. This can be explained from the tight

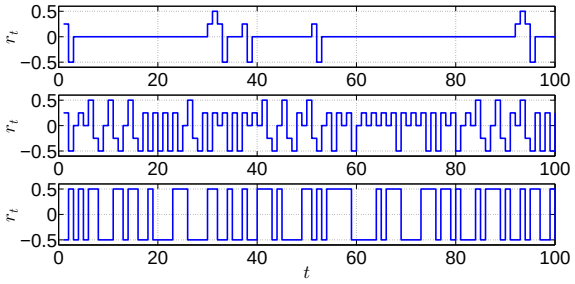

Fig. 2. Part of the reference signal $\{r_t\}_{t=1}^{500}$. **Top**: Optimal reference signal. **Middle**: Optimal reference signal without probabilistic constraints. **Bottom**: Random binary signal.

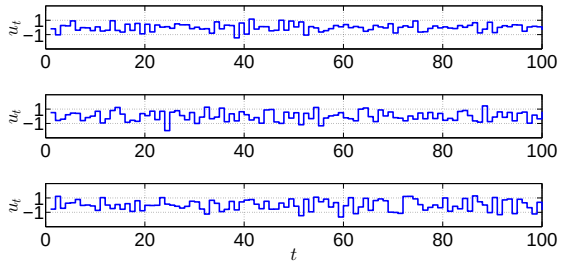

Fig. 3. Part of the input signal $\{u_t\}_{t=1}^{500}$. **Top**: Input signal for the optimal reference. **Middle**: Input signal for the optimal reference without probabilistic constraints. **Bottom**: Input signal for a random binary reference.

probabilistic bounds imposed for $\{u_t\}$, which restricts the excitation provided by $\{r_t\}$. If we compare the previous signal with the one obtained by solving (17) without probabilistic bounds, we see that the reference signal contains more oscillations when the probabilistic bounds are removed.

Figures 3 and 4 present one realization for the resulting input $\{u_t\}_{t=1}^{500}$ and output $\{y_t\}_{t=1}^{500}$, respectively. From those realizations, we conclude that, for the optimal reference, the input and output are inside the limiting regions 93.8%, and 96% of the time, respectively, which satisfies the design requirements. On the other hand, for the reference signal obtained by solving (17) without probabilistic bounds, we have that the input and output satisfies the constraints 86.6% and 93.4% of the time, respectively. Therefore, in this example we need to incorporate the probabilistic bounds to guarantee that both the input and output of the system are inside the desired region with the prescribed confidence level. With the previous modification, we restrict the set of optimal feasible solutions for the problem of minimum variance to the subset of optimal solutions satisfying the probabilistic bounds. Finally, for the random binary reference, we have that the input and output are inside the confidence region 90.8%, and 79.6% of the time, which does not satisfy the confidence bounds for the system.

To analyze the identification performance, Figure 5 presents the application ellipsoid for the parameter θ, together with the resulting identification ellipsoids and 50 identified parameters obtained with the optimal reference with probabilistic bounds, the optimal reference without probabilistic bounds, and for the random binary reference.

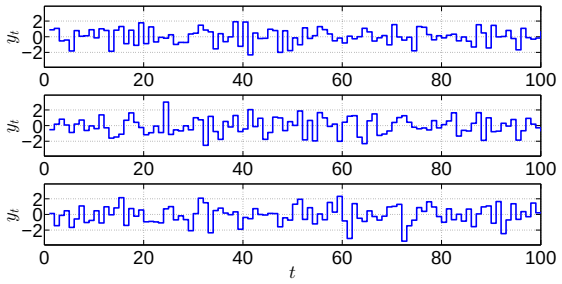

Fig. 4. Part of the output $\{y_t\}_{t=1}^{500}$. **Top**: Output signal for the optimal reference. **Middle**: Output signal for the optimal reference without probabilistic constraints. **Bottom**: Output signal for a random binary reference.

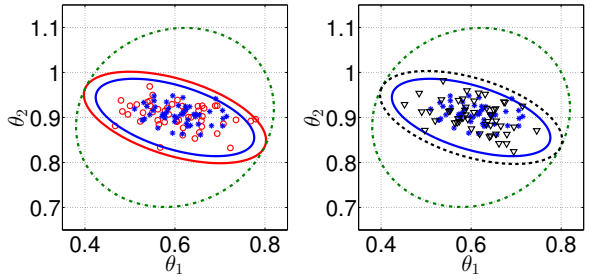

Fig. 5. Application ellipsoid (green, dot-dashed line) with the respective identification ellipsoids. **Blue, continuous line:** Identification ellipsoid for the random binary reference (realizations marked with $*$). **Red, continuous line:** Identification ellipsoid for the optimal reference with probabilistic bounds (realizations marked with circles). **Black, dashed line:** Identification ellipsoid for the optimal reference without probabilistic bounds (realizations marked with triangles).

From this figure we conclude that the 98% confidence level set for the identified parameters lies completely inside the application ellipsoid for all the reference signals. As expected, the confidence level set for the random binary reference is smaller than the ones obtained with the proposed technique, since the variance of this signal is greater than the one obtained with the optimal references. Hence, the random binary reference excites the system more than required, which makes the cost function in optimization problem (16) greater than the cost obtained with the proposed method. Indeed, the cost functions are $J^{\text{opt}} = 541.6$ for the optimal experiment with probabilistic bounds, and $J^{binary} = 695.8$ for a random binary reference, which is in line with the size of the uncertainty ellipsoids in Figure 5. On the other hand, we see that the confidence ellipsoids for the estimated parameters are almost the same when a reference signal is designed by including or excluding the probabilistic bounds on the input and output.

V. Conclusion

In this work a method to design input sequences for closed-loop experiments has been proposed. The method considers the input sequence as a realization of a stationary process minimizing the experimental cost, and subject to performance constraints. Using elements from graph-theory, the elements in the set of stationary processes are described as a convex combination of the measures associated with the prime cycles of the corresponding Brujin graph. Therefore, both experimental cost and constraints can be expressed as a convex combination of the values associated with the extreme measures in the set, which are computed using Monte-Carlo methods. An interesting feature of this approach is that probabilistic constraints become convex in the decision variables. The numerical example shows that this approach is an attractive method for the design of input sequences to identify models in a closed-loop setting.

Future work in the subject will be focused on extending the analysis to more general model structures including non-linear state space models. More extensions could be robust and adaptive approaches to remove the assumptions on the knowledge of the true system, as explained in Remark 1.

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