Optimal tracking performance for unstable Tall and Squared-up plant models

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***Abstract*— This article is focused on the best achievable tracking performance of unstable tall and squared-up plant models. The squared-up plant is originated by adding control inputs to a tall system to become a square plant. The work is developed for discrete time, LTI systems, when a decaying signal is considered as reference. Closed form expressions for the best tracking performance for one and two degree of freedom control schemes are presented, and a case study of the benefits of adding control inputs is also considered.**

***Index Terms*— Performance bounds, two degree of freedom control, augmented systems, optimal control, multivariable control.**

I. INTRODUCTION

This work is focused on the computation of performance bounds in discrete time MIMO feedback control systems. A performance bound describes the best achievable performance, measured by a specific cost function, which can be achieved in the control of a plant. This index can be employed to establish a benchmark against which the result of any design procedure can be compared.

Performance bounds in control systems have been of interest in the last decade. In this period, significant results have been obtained in this field (see, e.g, [1], [2], [3], [4], and the references therein). The main contribution of these works is the development of closed form expressions for the best achievable performance, when a feedback control system is considered. In [1] the best achievable performance for continuous time feedback control system is studied. The results in [1] suggest that unstable poles, non-minimum-phase zeros and time delays worsen the optimal tracking performance. Similar results are presented in [3], extending the analysis to discrete time MIMO feedback control systems. However, these results are only useful for particular delay structures, and for square systems.

Plants with general delay structures are studied in [5]. In that work, the best achievable tracking performance is computed, when stable discrete time MIMO control systems are considered. In the same spirit than [3], the results in [5] shows that finite and infinite non-minimun-phase zeros have a deleterious effect on tracking performance.

The results presented in [1], [3], [5] can be only applied to square systems. Results for tall plants have been reported in [2], [6], [7], [8]. A key issue in the control of tall plants is they are non-right invertible. In [2] the best achievable tracking performance for SIMO systems is computed. The results in [2] shows that not only non-minimum phase zeros and unstable poles affect the optimal performance, but also the total variation of the plant direction with frequency. Similar results are presented in [4], where the closed form expressions are also derived for discrete time SIMO systems, and an unify approach for continuous and discrete time systems is also discussed. Nevertheless, the assumptions on the reference vector make the results applicable only to special situations. A related work [9], by the same authors, deals with the regulation problem for tall plants, that is, the reference was assumed to be a vector Kronecker delta. The results in [9] shows that the control performance is improved when additional channels are added.

In despite of the recent results for SIMO plants, the optimal performance reached by them is limited, since the number of control channels is less than the number of outputs. One method to improve the performance of control for tall systems is by adding control inputs. As a first contribution of this paper, we propose a methodology to quantify the benefits of adding control inputs to a tall system, to make it square. This approach selects a cost function for the case when the reference is a decaying signal. As a second contribution, we compute a closed form expression for the best achievable performance for unstable tall and squared-up systems. Finally, we study the benefits of adding control inputs in unstable tall plants through specific cases.

The remainder of this paper is as follows. Section II introduces notation and preliminaries of this work; Section III presents the best achievable tracking performance for one degree of freedom control scheme; Section IV studies the best achievable tracking performance in two degree of freedom feedback control; Section V presents a study of the benefits of adding control inputs to an unstable tall system; finally, Section VI presents conclusions of this work and future research.

II. NOTATION AND PRELIMINARIES

In the current work the following notation is used: $\mathbb{C}$ is the complex set, $\mathbb{C}^{n\times m}$ is the complex set of $n\times m$ matrices, $\mathbb{R}$ is the real set, and $\mathbb{R}^{n\times m}$ is the real set of $n\times m$ matrices. Given a matrix $\mathbf{A}\in\mathbb{C}^{n\times m}$, $\mathbf{A}^T$ and $\mathbf{A}^H$ define its transpose and complex conjugate transpose, respectively. For a complex number x, $\overline{x}$ and $|x|$ are defined as its conjugate and magnitude, respectively; $\mathcal{R}_p^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational and proper; $\mathcal{R}_{sp}^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational and strictly proper; $\mathcal{RH}_\infty^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational, stable and proper; $\mathcal{RH}_2^{n\times m}$ is the set of $n\times m$ transfer matrices which are real rational, stable

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and strictly proper, and $\mathcal{RH}_2^{\perp\, n\times m}$ the set of $n \times m$ transfer matrices which are constant, improper and/or unstable. Given a function $\mathbf{X}[z] \in \mathbb{C}^{n\times m}$, $\mathbf{X}[z]^{\sim}$ is defined as

$$\mathbf{X}[z]^{\sim} \triangleq \mathbf{X}\left[\overline{z}^{-1}\right]^{H} , \tag{1}$$

which is reduced to $\mathbf{X}[z]^{\sim} = \mathbf{X}\left[z^{-1}\right]^{T}$ for real rational case.

A transfer matrix $\mathbf{U}[z] \in \mathbb{C}^{n\times m}$ ($n \geq m$) is unitary if and only if

$$\mathbf{U}[z]^{\sim}\, \mathbf{U}[z] = \mathbf{I_m} , \tag{2}$$

where $\mathbf{I_k}$ is the $k \times k$ identity matrix.

A transfer matrix $\mathbf{P}[z] \in \mathcal{RH}_\infty^{n\times m}$ admits an inner-outer factorization

$$\mathbf{P}[z] = \mathbf{P_i}[z]\; \mathbf{P_o}[z] , \tag{3}$$

where $\mathbf{P_i}[z] \in \mathcal{RH}_\infty^{n\times m}$ is an inner factor, i.e., $\mathbf{P_i}[z]^{\sim}\, \mathbf{P_i}[z] = \mathbf{I_m}$. On the other hand, $\mathbf{P_o}[z] \in \mathcal{RH}_\infty^{m\times m}$ is an outer factor, and it is right invertible, which is analytical on $|z| > 1$ [10].

A number $c \in \mathbb{C}$ is said to be a zero of $\mathbf{P}[z] \in \mathcal{R}_{sp}^{n\times m}$ if and only if $\text{rank}\{\mathbf{P}[c]\} < \text{normal\_rank}\{\mathbf{P}[z]\}$. If $|c| > 1$, c is called a non-minimum-phase (NMP) zero, otherwise c is called a minimum-phase (MP) zero. For a square plant $\mathbf{H}[z] \in \mathcal{RH}_2^{n\times n}$,

$$\mathbf{H}[z] = \mathbf{E_{I,dc}}[z]\; \mathbf{H_{FM}}[z] , \tag{4}$$

where $\mathbf{H_{FM}}[z] \in \mathcal{RH}_\infty^{n\times n}$ is a minimum-phase (MP) transfer matrix, and $\mathbf{E_{I,dc}}[z] \in \mathcal{RH}_2^{n\times n}$ is a unitary left non-minimum-phase (NMP) zeros interactor for $\mathbf{H}[z]$, which is defined as

$$\mathbf{E_{I,dc}}[z]^{-1} \triangleq \mathbf{E_{I,c}}[z]^{-1}\; \mathbf{E_{I,d}}[z]^{-1} , \tag{5}$$

where $\mathbf{E_{I,c}}[z]^{-1} \in \mathcal{RH}_2^{\perp\, n\times n}$ and $\mathbf{E_{I,d}}[z]^{-1} \in \mathcal{RH}_2^{\perp\, n\times n}$ are the unitary left finite and infinite NMP zeros interactor, respectively,

$$\mathbf{E_{I,c}}[z]^{-1} \triangleq \prod_{k=1}^{n_c} \left\{ \frac{1-c_k}{1-\overline{c_k}}\, \frac{1-z\,\overline{c_k}}{z-c_k}\, \boldsymbol{\eta}_k \boldsymbol{\eta}_k^H + \mathbf{U}_k \mathbf{U}_k^H \right\} , \tag{6}$$

$$\mathbf{E_{I,d}}[z]^{-1} \triangleq \prod_{k=1}^{n_z} \left\{ z\, \boldsymbol{\eta}_{\infty_k} \boldsymbol{\eta}_{\infty_k}^H + \mathbf{U}_{\infty_k} \mathbf{U}_{\infty_k}^H \right\} , \tag{7}$$

and $\boldsymbol{\eta}_k$, $\boldsymbol{\eta}_{\infty_k} \in \mathbb{C}^{n\times 1}$ and $\mathbf{U}_k$, $\mathbf{U}_{\infty_k} \in \mathbb{C}^{n\times(n-1)}$ satisfy $\boldsymbol{\eta}_k \boldsymbol{\eta}_k^H + \mathbf{U}_k \mathbf{U}_k^H = \boldsymbol{\eta}_{\infty_k} \boldsymbol{\eta}_{\infty_k}^H + \mathbf{U}_{\infty_k} \mathbf{U}_{\infty_k}^H = \mathbf{I_n}$, $\forall k$ [5].

The expectation operator is denoted by $\mathcal{E}\{\cdot\}$. The 2 norm for a system $\mathbf{B}[z] \in \mathbb{C}^{n\times m}$ no singular on $|z| = 1$ is defined as

$$\|\mathbf{B}[z]\|_2 = \sqrt{\text{trace}\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{B}\left[e^{j\omega}\right]^H \mathbf{B}\left[e^{j\omega}\right] d\omega \right\}} . \tag{8}$$

Under the 2 norm, $\mathcal{RH}_2^{n\times m}$ and $\mathcal{RH}_2^{\perp\, n\times m}$ are orthogonal sets. Therefore,

$$\left\| \{\mathbf{A}[z]\}_{\mathcal{H}_2^\perp} + \{\mathbf{A}[z]\}_{\mathcal{H}_2} \right\|_2^2 = \left\| \{\mathbf{A}[z]\}_{\mathcal{H}_2^\perp} \right\|_2^2 + \left\| \{\mathbf{A}[z]\}_{\mathcal{H}_2} \right\|_2^2 . \tag{9}$$

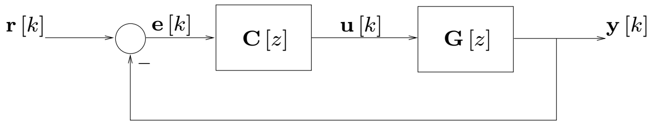

Fig. 1. One Degree of freedom control scheme.

where $\{\mathbf{A}[z]\}_{\mathcal{H}_2^\perp}$ denotes the part of $\mathbf{A}[z] \in \mathcal{RH}_2^{\perp\, n\times m}$, and $\{\mathbf{A}[z]\}_{\mathcal{H}_2}$ denotes the part of $\mathbf{A}[z] \in \mathcal{RH}_2^{n\times m}$.

III. Tracking performance bounds for perfect and delayed control channels

This section is focused on the best tracking performance for tall and squared-up plant models, assuming that the control channels are either perfect or they only have propagation delays, and when a decaying signal is applied as closed loop reference. As first part of this section, the problem is stated, and then closed form expressions for the best tracking performance are derived for unstable plant models.

A. Problem formulation

Consider the 1-dof[1] control scheme, depicted in Figure 1. In that figure, $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$ ($n \geq m$) is the plant model, $\mathbf{C}[z] \in \mathcal{R}_p^{m\times n}$ is a feedback controller, and $\mathbf{r}[k] \in \mathbb{R}^n$, $\mathbf{e}[k] \in \mathbb{R}^n$, $\mathbf{u}[k] \in \mathbb{R}^m$ are the reference, tracking error and control signals, respectively. In this section, we study the functional

$$J \triangleq \sum_{k=0}^{\infty} \mathbf{e}[k]^T\, \mathbf{e}[k] , \tag{10}$$

when $\mathbf{r}[k] \triangleq \boldsymbol{\nu}\, \lambda^k$, with $\boldsymbol{\nu} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, such that $|\lambda| < 1$. This reference guarantees the convergence of J to a finite value when $n > m$.

Assuming the closed loop internally stable, then

$$J = \|\mathbf{E}[z]\|_2^2 , \tag{11}$$

where $\mathbf{E}[z]$ is the $\mathcal{Z}$-Transform of the tracking error $\mathbf{e}[k]$. Using the closed loop description given in Figure 1, then J becomes

$$J = \left\| \left(\mathbf{I_n} + \mathbf{G}[z]\,\mathbf{C}[z]\right)^{-1} \frac{\boldsymbol{\nu}}{z-\lambda} \right\|_2^2 , \tag{12}$$

where the $\mathcal{Z}$-Transform of the reference $\mathbf{r}[k]$ is used.

The expression (12) is non-linear in the controller $\mathbf{C}[z]$. To solve this problem, we use the coprime factorization of $\mathbf{G}[z]$, defined as [10]

$$\mathbf{G}[z] \triangleq \mathbf{N_D}[z]\,\mathbf{D_D}[z]^{-1} = \mathbf{D_I}[z]^{-1}\,\mathbf{N_I}[z] , \tag{13}$$

with $\mathbf{N_I}[z], \mathbf{N_D}[z] \in \mathcal{RH}_\infty^{n\times m}$, $\mathbf{D_I}[z] \in \mathcal{RH}_\infty^{n\times n}$ and $\mathbf{D_D}[z] \in \mathcal{RH}_\infty^{m\times m}$. These factors can be used to describe a stabilizing controller $\mathbf{C}[z]$ as

$$\mathbf{C}[z] \triangleq \left(\mathbf{Y_D}[z] - \mathbf{D_D}[z]\,\mathbf{Q}[z]\right)\left(\mathbf{N_D}[z]\,\mathbf{Q}[z] - \mathbf{X_D}[z]\right)^{-1}$$

[1] Degree of freedom.

$$= \left(\mathbf{Q}[z]\,\mathbf{N_I}[z] - \mathbf{X_I}[z]\right)^{-1}\left(\mathbf{Y_I}[z] - \mathbf{Q}[z]\,\mathbf{D_I}[z]\right), \quad (14)$$

where $\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}$ is the design parameter, and $\mathbf{Y_I}[z], \mathbf{Y_D}[z] \in \mathcal{RH}_\infty^{m\times n}$, $\mathbf{X_I}[z] \in \mathcal{RH}_\infty^{m\times m}$, $\mathbf{X_D}[z] \in \mathcal{RH}_\infty^{n\times n}$ are such that they satisfy the double Bezout identity

$$\begin{bmatrix} \mathbf{X_I}[z] & -\mathbf{Y_I}[z] \\ -\mathbf{N_I}[z] & \mathbf{D_I}[z] \end{bmatrix}\begin{bmatrix} \mathbf{D_D}[z] & \mathbf{Y_D}[z] \\ \mathbf{N_D}[z] & \mathbf{X_D}[z] \end{bmatrix} = \begin{bmatrix} \mathbf{I_m} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_n} \end{bmatrix}. \quad (15)$$

Using (14) into (12), we get

$$J = \left\| \left(\mathbf{X_D}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\right)\mathbf{D_I}[z]\ \frac{\boldsymbol{\nu}}{z-\lambda} \right\|_2^2. \quad (16)$$

To simplify our analysis, we make the following assumption:

Assumption 1: $\boldsymbol{\nu} \in \mathbb{R}^n$ is a random vector that satisfies

$$\mathcal{E}\{\boldsymbol{\nu}\} = \mathbf{0}, \quad (17)$$

$$\mathcal{E}\left\{\boldsymbol{\nu}\boldsymbol{\nu}^T\right\} = \mathbf{I_n}. \quad (18)$$

Taking the expectation to (16) and under Assumption 1, we can write $\mathcal{E}\{J\}$ as

$$\mathcal{E}\{J\} = \left\| \left(\mathbf{X_D}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\right)\mathbf{D_I}[z]\ \frac{1}{z-\lambda} \right\|_2^2. \quad (19)$$

Finally, our study is focused on minimizing (19), subject to $\mathbf{Q}[z] \in \mathcal{RH}_\infty^{m\times n}$. This can be written as

Problem 1: Given $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$, $n \geq m$, find

$$J^{\rm opt} \triangleq \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{X_D}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\right)\mathbf{D_I}[z]\ \frac{1}{z-\lambda} \right\|_2^2, \quad (20)$$

and, if $J^{\rm opt}$ is achievable, find $\mathbf{Q^{opt}}[z]$ stable and proper that achieves $J^{\rm opt}$.

Problem 1 will be solved in the next section.

B. Optimal tracking performance: unstable plants

When the plants are unstable, we can obtain closed form expressions for the optimal tracking performance, when an exponentially decaying reference is considered.

Theorem 1: Consider the Problem 1, a plant $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$, $n \geq m$, with n_p different unstable poles, and its coprime factorization given by (13). Also, define an inner-outer factorization of $\mathbf{N_D}[z]$ as $\mathbf{N_D}[z] \triangleq \mathbf{N_{Di}}[z]\,\mathbf{N_{Do}}[z]$, and a factorization of $\mathbf{D_I}[z]$ given by $\mathbf{D_I}[z] \triangleq \mathbf{D_{I,FM}}[z]\,\mathbf{E_{D,c}}[z]$, with $\mathbf{E_{D,c}}[z]$ defined as (6), and $\mathbf{D_{I,FM}}[z]$ is stable, biproper and MP. Then, the solution of the Problem 1 is achieved by choosing $\mathbf{Q}[z] \triangleq \mathbf{Q^{opt}}[z]$, where

$$\mathbf{Q^{opt}}[z] \triangleq \arg \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \mathcal{E}\{J\}$$
$$= \mathbf{N_{Do}}[z]^{-1}\left(\mathbf{P}_1[\lambda] + \mathbf{P}_2[z]\right)\mathbf{D_{I,FM}}[z]^{-1}, \quad (21)$$

and

$$\mathbf{P}_1[z] \triangleq \sum_{i=1}^{n_p} \frac{\mathbf{A}_i}{z-p_i}, \quad (22)$$

$$\mathbf{P}_2[z] \triangleq \mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1} - \mathbf{P}_1[z], \quad (23)$$

$$\mathbf{P}[z] \triangleq \mathbf{N_{Di}}\left[\lambda^{-1}\right]^T + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z], \quad (24)$$

$$\mathbf{A}_i \triangleq \lim_{z\to p_i} (z-p_i)\,\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}, \quad (25)$$

and $J^{\rm opt}$ is given by

$$J^{\rm opt} = J_s^{\rm opt} + J_u^{\rm opt}, \quad (26)$$

where

$$J_s^{\rm opt} \triangleq \frac{1}{1-\lambda^2}\left(n - \text{trace}\left\{\mathbf{N_{Di}}\left[\lambda^{-1}\right]\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right\}\right), \quad (27)$$

$$J_u^{\rm opt} \triangleq \frac{1}{1-\lambda^2}\text{trace}\left\{\left(\mathbf{P}_1\left[\lambda^{-1}\right] - \mathbf{P}_1[\lambda]\right)^T\mathbf{P}_1[\lambda]\right\} + \text{trace}\left\{\sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\overline{p_i}\cdot\frac{\mathbf{A}_i^H\mathbf{A}_j}{(1-\lambda\overline{p_i})(\overline{p_i}-\lambda)(\overline{p_i}p_j-1)}\right\}. \quad (28)$$

Proof: First, note from (15) that $\mathbf{N_D}[z]\,\mathbf{Y_I}[z] + \mathbf{I_n} = \mathbf{X_D}[z]\,\mathbf{D_I}[z]$. Then, (20) becomes

$$J^{\rm opt} = \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{I_n} + \mathbf{N_D}[z]\,\mathbf{Y_I}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda} \right\|_2^2. \quad (29)$$

Using the unitary matrix

$$\boldsymbol{\Lambda}[z] \triangleq \begin{bmatrix} \mathbf{N_{Di}}[z]^\sim \\ \mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim \end{bmatrix}, \quad (30)$$

into (29), we can write

$$J^{\rm opt} = \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \left\| \boldsymbol{\Lambda}[z]\left(\mathbf{I_n} + \mathbf{N_D}[z]\,\mathbf{Y_I}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda} \right\|_2^2$$
$$= \left\| \left(\mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim\right)\frac{1}{z-\lambda} \right\|_2^2 + \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}} \left\| \left(\mathbf{N_{Di}}[z]^\sim + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda} \right\|_2^2. \quad (31)$$

Also, noting that

$$\left(\mathbf{N_{Di}}[z]^\sim - \mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right)\frac{1}{z-\lambda} \in \mathcal{RH}_2^{\perp\, m\times n} \quad (32)$$

$$\left(\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T + \mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z] - \mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda} \in \mathcal{RH}_2^{m\times n}, \quad (33)$$

it is clear that

$$J^{\rm opt} = \left\| \left(\mathbf{I_n} - \mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^\sim\right)\frac{1}{z-\lambda} \right\|_2^2 +$$

$$+\left\|\left(\mathbf{N_{Di}}[z]^{\sim}-\mathbf{N_{Di}}\left[\lambda^{-1}\right]^{T}\right)\frac{1}{z-\lambda}\right\|_2^2$$
$$+\inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}}\left\|\left(\mathbf{N_{Di}}\left[\lambda^{-1}\right]^{T}+\mathbf{N_{Do}}[z]\,\mathbf{Y_I}[z]\right.\right.$$
$$\left.\left.-\mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda}\right\|_2^2. \quad (34)$$

It is straightforward to prove that

$$\left\|\left(\mathbf{I_n}-\mathbf{N_{Di}}[z]\,\mathbf{N_{Di}}[z]^{\sim}\right)\frac{1}{z-\lambda}\right\|_2^2+\left\|\left(\mathbf{N_{Di}}[z]^{\sim}-\mathbf{N_{Di}}\left[\lambda^{-1}\right]^{T}\right)\frac{1}{z-\lambda}\right\|_2^2=$$
$$\frac{1}{1-\lambda^2}\left(n-\mathrm{trace}\left\{\mathbf{N_{Di}}\left[\lambda^{-1}\right]\mathbf{N_{Di}}\left[\lambda^{-1}\right]^{T}\right\}\right)=J_s^{\mathrm{opt}}, \quad (35)$$

and thus (34) becomes

$$J^{\mathrm{opt}}=J_s^{\mathrm{opt}}+\inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}}\left\|\left(\mathbf{P}[z]-\mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_I}[z]\right)\frac{1}{z-\lambda}\right\|_2^2, \quad (36)$$

with $\mathbf{P}[z]$ defined by (22). On the other hand, if we consider the factorization $\mathbf{D_I}[z]\triangleq\mathbf{D_{I,FM}}[z]\,\mathbf{E_{D,c}}[z]$, where $\mathbf{D_{I,FM}}[z]$ is biproper, stable and MP, and $\mathbf{E_{D,c}}[z]$ is unitary, then

$$J^{\mathrm{opt}}=J_s^{\mathrm{opt}}+\inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}}\left\|\left(\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}\right.\right.$$
$$\left.\left.-\mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_{I,FM}}[z]\right)\frac{1}{z-\lambda}\right\|_2^2. \quad (37)$$

The expression $\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}$ contains stable and unstable terms. Therefore, it is necessary to make a partial fraction expansion. Given that the unstable terms are the n_p unstable plant poles (with multiplicity one), then

$$\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}\triangleq\mathbf{P}_1[z]+\mathbf{P}_2[z], \quad (38)$$

where

$$\mathbf{P}_1[z]\triangleq\sum_{i=1}^{n_p}\frac{\mathbf{A}_i}{z-p_i}, \quad (39)$$
$$\mathbf{P}_2[z]\triangleq\mathbf{P}[z]\,\mathbf{E_{D,c}}[z]^{-1}-\mathbf{P}_1[z], \quad (40)$$

with $\mathbf{A}_i$ defined as (23). According to the definitions given in (22) and (23), it follows that $\mathbf{P}_1[z]\in\mathcal{RH}_2^{\perp\,m\times n}$ and $\mathbf{P}_2[z]\in\mathcal{RH}_\infty^{m\times n}$. Therefore,

$$J^{\mathrm{opt}}=J_s^{\mathrm{opt}}+\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2$$
$$+\inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}}\left\|\left(\mathbf{P}_1[\lambda]+\mathbf{P}_2[z]\right.\right.$$
$$\left.\left.-\mathbf{N_{Do}}[z]\,\mathbf{Q}[z]\,\mathbf{D_{I,FM}}[z]\right)\frac{1}{z-\lambda}\right\|_2^2. \quad (41)$$

We can then note from (41) that $\mathbf{Q}[z]$ can be chosen according to (21) to obtain

$$J^{\mathrm{opt}}\triangleq J_s^{\mathrm{opt}}+\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2. \quad (42)$$

To complete this proof, we need to prove that the second term in the right hand side of the equality in (42) is J_u^{opt}, with J_u^{opt} defined as (28). For this purpose, consider

$$\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2=\mathrm{trace}\left\{\frac{1}{2\pi j}\oint\frac{\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)^{\sim}}{1-z\lambda}\times\frac{\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)}{z-\lambda}dz\right\}, \quad (43)$$

where the integral is on $|z|=1$, counterclockwise oriented. The expression (43) can be computed using the Cauchy's Residue Theorem [11]. We thus obtain

$$\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2=\frac{1}{1-\lambda^2}\mathrm{trace}\left\{\mathbf{P}_1[\lambda]^T\,\mathbf{P}_1[\lambda]\right\}$$
$$+\mathrm{trace}\left\{\frac{1}{2\pi j}\oint\left[\frac{\mathbf{P}_1[z]^{\sim}\,\mathbf{P}_1[z]-\mathbf{P}_1[z]^{\sim}\,\mathbf{P}_1[\lambda]}{(1-z\lambda)(z-\lambda)}\frac{-\mathbf{P}_1[\lambda]^T\,\mathbf{P}_1[z]}{(1-z\lambda)(z-\lambda)}\right]dz\right\}, \quad (44)$$

which can be rewritten as

$$\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2=\mathrm{trace}\left\{\frac{1}{2\pi j}\oint\frac{\mathbf{P}_1[z]^{\sim}\,\mathbf{P}_1[z]-\mathbf{P}_1[z]^{\sim}\,\mathbf{P}_1[\lambda]}{(1-z\lambda)(z-\lambda)}dz\right\}$$
$$=-\frac{1}{1-\lambda^2}\mathrm{trace}\left\{\mathbf{P}_1[\lambda]^T\,\mathbf{P}_1[\lambda]\right\}+\mathrm{trace}\left\{\frac{1}{2\pi j}\oint\frac{\mathbf{P}_1[z]^{\sim}\,\mathbf{P}_1[z]}{(1-z\lambda)(z-\lambda)}dz\right\}. \quad (45)$$

The term inside the integral in (45) has, at least, two NMP at infinity. This allows one to use the result reported in [12] to compute (45) as

$$\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2=\frac{1}{1-\lambda^2}\mathrm{trace}\left\{\left(\mathbf{P}_1\left[\lambda^{-1}\right]-\mathbf{P}_1[\lambda]\right)^T\mathbf{P}_1[\lambda]\right\}$$
$$+\mathrm{trace}\left\{\sum_{i=1}^{n_p}\operatorname*{Res}_{z=\overline{p_i}^{-1}}\left\{\frac{\mathbf{P}_1[z]^{\sim}\,\mathbf{P}_1[z]}{(1-z\lambda)(z-\lambda)}\right\}\right\}, \quad (46)$$

and, therefore,

$$\left\|\left(\mathbf{P}_1[z]-\mathbf{P}_1[\lambda]\right)\frac{1}{z-\lambda}\right\|_2^2=\frac{1}{1-\lambda^2}\mathrm{trace}\left\{\left(\mathbf{P}_1\left[\lambda^{-1}\right]-\mathbf{P}_1[\lambda]\right)^T\mathbf{P}_1[\lambda]\right\}$$

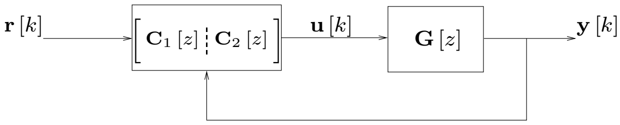

Fig. 2. Two Degree of freedom control scheme.

$$+\operatorname{trace}\left\{\sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\overline{p_i}\cdot\frac{\mathbf{A}_i^H\mathbf{A}_j}{(1-\lambda\overline{p_i})(\overline{p_i}-\lambda)(\overline{p_i}\,p_j-1)}\right\}. \tag{47}$$

Using (47) into (42) we obtain (26), concluding this proof. ■

The result presented in Theorem 1 shows that the optimal tracking performance for an unstable plant can be analysed by parts. The first term in (26) is a function of the finite and the infinite NMP zeros of $\mathbf{G}[z]$. On the other hand, the second term in (26) is a function of the unstable poles of $\mathbf{G}[z]$. The term J_u^{opt} shows that, when $\lambda \to \overline{p}_i^{-1}$, the index goes to infinity. This can be understood when we consider Assumption 1, because this assumption takes an average over all possible values of the reference vector. Finally, we must note that J^{opt} depends explicitly on the reference parameter λ, and on the number of output channels n.

Note that we can simplify Theorem 1 when $n = m$, i.e., we consider an unstable square system. In this case, we can use an explicit form for the inner factor $\mathbf{N_{Di}}[z]$, given by the left unitary NMP zeros interactor of $\mathbf{N_D}[z]$, defined as (5)–(7).

The results presented in this section are obtained by assuming that the closed loop system has only 1 design parameter. The next section studies the optimal tracking performance when a 2 parameter controller scheme is used in a closed loop configuration.

IV. Tracking performance bounds for perfect and delayed control channels: 2-dof control loop

In this section, we study the optimal tracking performance in a 2-dof control loop, when an exponentially decaying reference is used. This section will allow to discuss the benefits of adding control inputs, when the closed loop has 2 parameters to be designed.

A. Problem formulation

Consider the 2-dof control loop depicted in Figure 2. In this scheme, $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$ ($n \geq m$) is the plant model, $\mathbf{r}[k] \in \mathbb{R}^n$ is the reference, $\mathbf{y}[k] \in \mathbb{R}^n$ is the system output, and $\mathbf{u}[k] \in \mathbb{R}^m$ is the control signal, defined as

$$\mathbf{U}[z] \triangleq \mathbf{C}_1[z]\,\mathbf{R}[z] + \mathbf{C}_2[z]\,\mathbf{Y}[z]\,, \tag{48}$$

where $\mathbf{C}_1[z] \in \mathcal{RH}_\infty^{m\times n}$, $\mathbf{C}_2[z] \in \mathcal{R}_p^{m\times n}$, and $\mathbf{R}[z], \mathbf{Y}[z], \mathbf{U}[z]$ are the $\mathcal{Z}$-Transforms of $\mathbf{r}[k], \mathbf{y}[k]$, and $\mathbf{u}[k]$, respectively. In this control configuration, the performance will be measured by

$$J \triangleq \sum_{k=0}^{\infty}(\mathbf{r}[k]-\mathbf{y}[k])^T(\mathbf{r}[k]-\mathbf{y}[k])\,, \tag{49}$$

when $\mathbf{r}[k] \triangleq \boldsymbol{\nu}\lambda^k$, $\boldsymbol{\nu} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $|\lambda| < 1$. Using Parseval's Theorem, (49) can be written as

$$J = \|\mathbf{R}[z] - \mathbf{Y}[z]\|_2^2\,. \tag{50}$$

The closed loop transfer matrices are non linear on parameters $\mathbf{C}_1[z]$ and $\mathbf{C}_2[z]$. Therefore, we use the double coprime factorization for $\mathbf{G}[z]$, given by

$$\mathbf{G}[z] \triangleq \mathbf{N_D}[z]\,\mathbf{D_D}[z]^{-1} = \mathbf{D_I}[z]^{-1}\,\mathbf{N_I}[z]\,, \tag{51}$$

where $\mathbf{N_I}[z], \mathbf{N_D}[z] \in \mathcal{RH}_\infty^{n\times m}$, $\mathbf{D_I}[z] \in \mathcal{RH}_\infty^{n\times n}$ and $\mathbf{D_D}[z] \in \mathcal{RH}_\infty^{n\times n}$ satisfy

$$\begin{bmatrix}\mathbf{X_I}[z] & -\mathbf{Y_I}[z]\\ -\mathbf{N_I}[z] & \mathbf{D_I}[z]\end{bmatrix}\begin{bmatrix}\mathbf{D_D}[z] & \mathbf{Y_D}[z]\\ \mathbf{N_D}[z] & \mathbf{X_D}[z]\end{bmatrix} = \begin{bmatrix}\mathbf{I_m} & \mathbf{0}\\ \mathbf{0} & \mathbf{I_n}\end{bmatrix}, \tag{52}$$

where $\mathbf{Y_I}[z], \mathbf{Y_D}[z] \in \mathcal{RH}_\infty^{m\times n}$, $\mathbf{X_I}[z] \in \mathcal{RH}_\infty^{m\times m}$ and $\mathbf{X_D}[z] \in \mathcal{RH}_\infty^{n\times n}$. The above definitions allow to build a stabilizing controller as [13]

$$\begin{aligned}\left[\mathbf{C}_1[z] \,\vdots\, \mathbf{C}_2[z]\right] &\triangleq (\mathbf{X_I}[z] - \mathbf{R}[z]\,\mathbf{N_I}[z])^{-1}\times\\ &\left[\mathbf{Q}[z] \,\vdots\, \mathbf{Y_I}[z] - \mathbf{R}[z]\,\mathbf{D_I}[z]\right]\,,\end{aligned} \tag{53}$$

with $\mathbf{Q}[z], \mathbf{R}[z] \in \mathcal{RH}_\infty^{m\times n}$. This factorization can be used to write J as

$$J = \left\|(\mathbf{I_n} - \mathbf{N_D}[z]\,\mathbf{Q}[z])\,\frac{\boldsymbol{\nu}}{z-\lambda}\right\|_2^2, \tag{54}$$

where the $\mathcal{Z}$-Transform of $\mathbf{r}[k]$ is used. If we assume that $\boldsymbol{\nu} \in \mathbb{R}^n$ satisfies Assumption 1, then $\mathcal{E}\{J\}$ becomes

$$\mathcal{E}\{J\} = \left\|(\mathbf{I_n}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z])\,\frac{1}{z-\lambda}\right\|_2^2. \tag{55}$$

The problem to be tackled in the coming section is stated next.

Problem 2: Given $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$, $n \geq m$, find

$$J^{\text{opt}} \triangleq \inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}}\left\|(\mathbf{I_n}[z] - \mathbf{N_D}[z]\,\mathbf{Q}[z])\,\frac{1}{z-\lambda}\right\|_2^2, \tag{56}$$

and, if J^{opt} is achievable, find $\mathbf{Q^{opt}}[z]$ stable and proper that achieves J^{opt}.

B. Optimal tracking performance: 2-dof control scheme

This section presents a closed form expression for the optimal tracking performance given in Problem 2, when a decaying reference is used:

Corollary 1: Consider the Problem 2, a plant $\mathbf{G}[z] \in \mathcal{R}_{sp}^{n\times m}$, $n \geq m$, with no zeros on $|z| = 1$, and its coprime factorization given by (51). Consider an inner-outer factorization for $\mathbf{N_D}[z]$, given by $\mathbf{N_D}[z] \triangleq \mathbf{N_{Di}}[z]\,\mathbf{N_{Do}}[z]$. Then, the solution of Problem 2 is achieved by choosing $\mathbf{Q}[z] \triangleq \mathbf{Q^{opt}}[z]$, where

$$\begin{aligned}\mathbf{Q^{opt}}[z] &\triangleq \arg\inf_{\mathbf{Q}[z]\in\mathcal{RH}_\infty^{m\times n}}\mathcal{E}\{J\}\\ &= \mathbf{N_{Do}}[z]^{-1}\,\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\,,\end{aligned} \tag{57}$$

and the optimal performance is given by

$$J^{\text{opt}} = \frac{1}{1-\lambda^2}\left(n - \text{trace}\left\{\mathbf{N_{Di}}\left[\lambda^{-1}\right]\mathbf{N_{Di}}\left[\lambda^{-1}\right]^T\right\}\right). \tag{58}$$

Proof: Direct from the proof of Theorem 1, replacing $\mathbf{Y_I}[z] = \mathbf{0}$ and $\mathbf{D_I}[z] = \mathbf{I_n}$. ■

The result presented in Corollary 1 can be particularized to square plants, by using the left unitary NMP zeros interactor $\mathbf{E_{I,dc}}[z]$ as the inner factor $\mathbf{N_{Di}}[z]$.

The result given in Corollary 1 shows that the optimal tracking performance only depends on plant NMP zeros, which are included in $\mathbf{N_{Di}}[z]$. If we compare this results with those obtained in Section III, we can observe that the optimal tracking cost is smaller in a 2-dof control configuration than in a 1-dof control scheme, only when the plant is unstable. Indeed, when the plant is stable, the results (26) and (58) are equal, which is consistent with previous results [14], [15].

V. Case study

The previous results are presented to allow an analysis of the improvement achieved when we control an augmented system, which is obtained by adding new control inputs to a tall system. Conversely, these results can be used to study the effect of deleting control inputs of a square system to become a tall plant. This study will be focused in cases studies, which allow to compare the benefits (or deterioration) of adding (or deleting) control inputs, under different structures.

A. Effect of unstable poles on tracking performance improvement

Apart from NMP zeros, other system dynamics that affect tracking performance are the unstable poles. As we can find in the literature, the unstable poles worsen the tracking performance, when a 1-dof control scheme is considered. In this section , we study the improvement on tracking performance, when a new control channel is added to an unstable tall plant.

Example 1: Consider a tall plant $\mathbf{G_A}[z] \in \mathcal{R}_{sp}^{2\times 1}$ defined as

$$\mathbf{G_A}[z] \triangleq \begin{bmatrix} \dfrac{z-0.5}{z(z-p)} & \dfrac{z-0.5}{z(z-0.8)} \end{bmatrix}^T, \tag{59}$$

with $p \in \mathbb{R}$, such that $|p| > 1$. The system presented in (59) has 1 unstable pole in $z = p$, and 1 NMP zero at infinity. To improve the tracking performance of the system, we propose to add 1 control channel to get

$$\mathbf{G_S}[z] \triangleq \left[\begin{array}{c:c} \mathbf{G_A}[z] & \begin{array}{c} \dfrac{z-0.2}{z(z-0.6)} \\ \\ \dfrac{z-0.4}{z(z-0.6)} \end{array} \end{array}\right]. \tag{60}$$

The augmented system (60) has the same unstable pole in $z = p$, 2 NMP zeros at infinity and, possibly, 1 finite NMP zero.

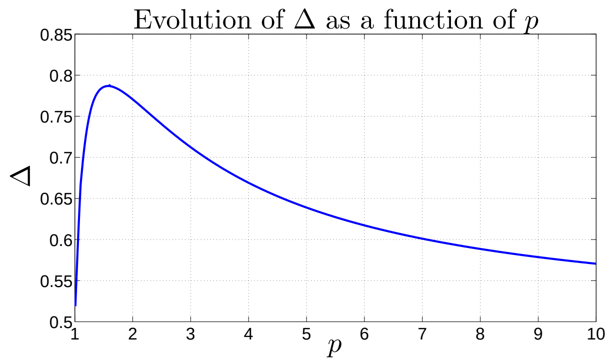

Fig. 3. Evolution of Δ as a function of p.

To study the benefits of adding this new control input, we consider a reference defined as $\mathbf{r}[k] \triangleq (0.9)^k \boldsymbol{\nu}$, with $k \in \mathbb{N}_0$ and $\boldsymbol{\nu} \in \mathbb{R}^2$. Defining J_A^{opt} as the optimal tracking performance of $\mathbf{G_A}[z]$, and J_S^{opt} as the optimal tracking performance of $\mathbf{G_S}[z]$, we study the evolution of Δ as a function of p, where

$$\Delta \triangleq \frac{J_A^{\text{opt}} - J_S^{\text{opt}}}{J_A^{\text{opt}}}. \tag{61}$$

Figure 3 shows the evolution of Δ as a function of the unstable pole p. We can observe that the benefits of adding a new control input are always over 50% for this case. In addition, these benefits are always nonzero, even if the square system has new finite NMP zeros, which worsen the tracking performance of the augmented system.

B. Effect of number of design parameters on tracking performance improvement

As we study in this paper, an additional degree of freedom improves the tracking performance of a control system. This effect is explored in the following example.

Example 2: Consider a tall plant $\mathbf{G_A}[z] \in \mathcal{R}_{sp}^{2\times 1}$ defined as

$$\mathbf{G_A}[z] \triangleq \begin{bmatrix} \dfrac{z-0.5}{z(z-p)} & \dfrac{z-0.5}{z(z-0.8)} \end{bmatrix}^T, \tag{62}$$

with $p \in \mathbb{R}$, such that $|p| > 1$. The system presented in (62) has 1 unstable pole in $z = p$, and 1 NMP zero at infinity. To improve the tracking performance of the system, we propose to add 1 control channel to get

$$\mathbf{G_S}[z] \triangleq \left[\begin{array}{c:c} \mathbf{G_A}[z] & \begin{array}{c} \dfrac{z-0.2}{z(z-0.6)} \\ \\ \dfrac{z-0.4}{z(z-0.6)} \end{array} \end{array}\right]. \tag{63}$$

The augmented system (63) has the same unstable pole in $z = p$, 2 NMP zeros at infinity and, possibly, 1 finite NMP zero. Thus, the tall and augmented systems are the same as we use in Example 1.

On the other hand, we define J_A^{opt} as the optimal tracking performance of the tall system in a 1-dof control scheme,

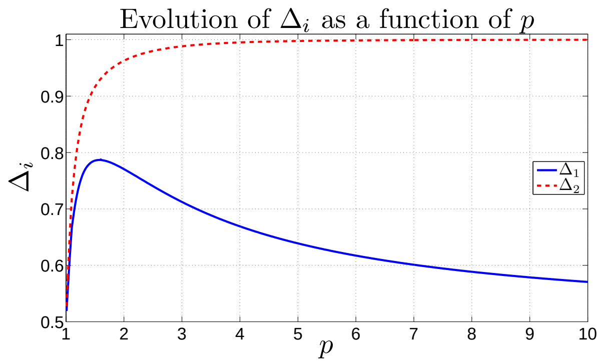

Fig. 4. Evolution of Δ_i as a function of p.

$J^{\rm opt}_{S,1}$ as the optimal tracking performance of the augmented system in a 1-dof control scheme, and $J^{\rm opt}_{S,2}$ as the optimal tracking performance of the augmented system in a 2-dof control architecture. To quantify the benefits of using the augmented system, we define

$$\Delta_i \triangleq \frac{J^{\rm opt}_{A} - J^{\rm opt}_{S,i}}{J^{\rm opt}_{A}}, \tag{64}$$

where $i \in \{1, 2\}$. This index allows to compare the benefits of adding a new control input in a closed loop system. Also, we can study the effect of a new design parameter in the system performance.

Figure 4 presents the evolution of Δ_i as a function of unstable pole p, for $i \in \{1, 2\}$. The results show that a new design parameter improves tracking performance better than 1-dof control scheme, when the same augmented system is used. Indeed, in a 2-dof control scheme, the unstable pole p does not appear into the optimal tracking performance; however, the optimal tracking performance in a 1-dof control scheme worsen as p increases.

VI. Conclusions

This paper presented the best achievable performance for tall and squared-up systems, when a decaying reference is considered. This reference guarantees the tracking error to have finite energy and, therefore, closed form expressions are computed for both tall and squared-up models. The results show that unstable poles and NMP zeros have a deleterious effect on the tracking performance in 1-dof control schemes.

To compare the performance with other control architectures, in this paper we also derive expressions for the best achievable tracking performance for 2-dof control schemes. As we can expect, the closed form expressions depend only on NMP zeros, which is consistent with known results [14], [15].

The previous results have been studied through specific examples. The expressions obtained in each case show that new control channels are beneficial for control purposes, when an initial unstable plant is considered. Moreover, the tracking performance is improved if two degrees of freedom are used.

Future work will be focused on study of benefits of using augmented systems when control channels with energy constraints are considered, and when data losses are presented in such channels.

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