Performance Bounds for SIMO and Squared-up Plant Models

Patricio E. Valenzuela, Mario E. Salgado and Eduardo I. Silva

***Abstract*— This paper presents performance bounds in the control of SIMO plant models, and the variation of those bounds when input channels are added to the plant so as to make it square. Our work focuses on discrete-time, linear time invariant (LTI) MIMO systems, minimizing the squared regulation error when an impulsive input disturbance is injected into the plant. Closed form expressions for the optimal regulation performance are developed, and the sensitivity of the performance to channel addition is also investigated. Finally, a numerical example is presented to illustrate the main results of the paper.**

***Index Terms*— SIMO systems, Performance bounds, Optimal control.**

I. INTRODUCTION

This work deals with the computation of performance bounds in the feedback control of MIMO plants. A performance bound describes the best achievable performance, measured by a specific cost function, that can be achieved in the control of a given plant, without constraints and without penalizing the control effort. When computing performance bounds, one aims at finding benchmarks against which the results of a design procedure can be compared.

Research on performance bounds and fundamental limitations is rooted in Bode's work [1], and has been the focus of much recent research (see, e.g., [2], [3], [4], [5], and the references therein). The contribution of these works has been to develop closed form characterizations of the best achievable performance for different set-ups. For instance, [3] writes the optimal tracking performance of MIMO systems for several references as a function of the unstable poles, non-minimum phase zeros, and their directions. These results apply only to right-invertible systems. Similar results are presented in [6], where stable right-invertible systems with arbitrary delay structure are considered.

Other results related to performance bounds on right-invertible systems have been presented in [7] and [8]. In these works, performance bounds subject to closed loop pole location constraints are computed.

In this paper we focus on SIMO plant models. A key problem in the control of such systems is that their transfer functions are not right-invertible. Performance bounds for SIMO systems have been reported in [9]. In that work, it is assumed that the reference direction belongs to the subspace spanned by the plant gain at zero frequency, and a closed form expression for the best tracking performance is derived. Similar results are presented in [10], when the plant has integrators. In that case, closed form expressions for the optimal tracking and regulation performances are presented, with hard constraints on the reference direction.

Other relevant results have been presented in [11]. In that work, regulation problems subject to impulsive input disturbances are addressed for SIMO plants that have only one zero at infinity. The approach in [11] might be extended to accommodate multiple zeros at infinity, but this is not explored in [11].

In despite of the recent work on SIMO systems, the performance reached by them is limited, because the degrees of freedom available to control is less than the number of system outputs. One method to improve the performance in SIMO systems is augmenting the number of control inputs available in the model of such plants. A review of existing methods to select system inputs and outputs is presented in [12], where such selection is discussed from a control perspective. However, these methods do not quantify the advantages of adding more control inputs to a system.

As a first contribution of this paper, we develop a closed form expression for the optimal regulation performance in the control of SIMO systems, assuming impulsive input disturbances. Our results are valid for plants that have non repeated finite non-minimum phase (NMP) zeros, but that are otherwise arbitrary. Thus, our results extend [11]. As a second contribution, we study tall plants, i.e. plants that have more outputs than inputs, which are augmented with additional control inputs and made square. In this case, we present a closed form characterization of the best achievable regulation performance for stable plants. As expected, our results show that additional control inputs are always useful to improve performance. We also illustrate our results using numerical simulations.

The remainder of this paper is organized as follow: Section II presents notation. Section III derives an expression for the the optimal performance in the control of SIMO systems. Section IV considers the case of tall plants that are squared up by adding a control channel. Section V presents an example, and Section VI draws conclusions.

II. NOTATION AND PRELIMINARIES

In this work, $\mathbb{C}$ denotes the complex numbers, $\mathbb{C}^{n\times m}$ denote the set of complex $n\times m$ matrices, $\mathbb{R}$ are the real numbers and $\mathbb{R}^{n\times m}$ represents the set of real $n\times m$ matrices. Given a matrix $A\in\mathbb{C}^{n\times m}$, A^T and A^H denotes its transpose and conjugate transpose respectively. If $x\in\mathbb{C}$, then $\overline{x}$ and $|x|$ denote its conjugate and magnitude, respectively. $\mathbb{R}\mathscr{H}_\infty$ is the set of all transfer functions (scalar or matrix) which are real rational, stable and proper. $\mathbb{R}\mathscr{H}_2$ is the set of all

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The authors are with the Departamento de Electrónica, Universidad Técnica Federico Santa María, Valparaíso, Chile. Email: mario.salgado@usm.cl

transfer functions (scalar or matrix) which are real rational, stable and strictly proper, and $\mathbb{R}\mathscr{H}_2^{\perp}$ is the set of all functions which are real rational and have no poles in $\{z \in \mathbb{C} : |z| < 1\}$. For a function $X[z] \in \mathbb{C}^{n\times m}$, we define $X[z]^{\sim} \triangleq X\left[\overline{z}^{-1}\right]^{H}$, which reduces to $X[z]^{\sim} \triangleq X\left[z^{-1}\right]^{T}$ in the real rational case. A matrix transfer function $U[z] \in \mathbb{C}^{n\times m}$ is said to be unitary iff $U[z]^{\sim} U[z] = I_m$, where I_m denotes the $m\times m$ identity matrix. All transfer functions $P[z] \in \mathbb{R}\mathscr{H}_\infty$ admits an inner-outer factorization $P[z] \triangleq \Phi_P[z]\,\Theta_P[z]$, where $\Phi_P[z] \in \mathbb{R}\mathscr{H}_\infty$ is an inner factor, and $\Theta_P[z] \in \mathbb{R}\mathscr{H}_\infty$ is right-invertible in $\mathbb{R}\mathscr{H}_\infty$ (an outer function). A number $c \in \mathbb{C}$ is a zero of $P[z] \in \mathbb{C}^{n\times m}$ iff $\operatorname{rank}\{P[c]\} <$ normal $\operatorname{rank}\{P[z]\}$. If $|c| < 1$, the zero is called minimum phase (MP), otherwise it is a Non-Minimum Phase (NMP) zero. We refer to the multiplicity of a NMP zero at infinity as the delay or the relative degree of $P[z]$. Any $P[z] \in \mathbb{C}^{n\times m}$, where $n \geq m$, can be factorized as $P[z] \triangleq \hat{P}[z]\,E_P^r[z]$, where $\hat{P}[z] \in \mathbb{C}^{n\times m}$ is real rational, stable, biproper and MP, and $E_P^r[z] \in \mathbb{C}^{m\times m}$ is a unitary function. Any stable $H[z] \in \mathbb{C}^{n\times m}$, where $m \geq n$, can be factorized as $H[z] \triangleq E_H^l[z]\,\hat{H}[z]$, where $\hat{H}[z] \in \mathbb{C}^{n\times m}$ is a real rational, stable, biproper and MP matrix transfer function, and $E_H^l[z] \in \mathbb{C}^{n\times n}$ is a left unitary interactor for $H[z]$, defined by $E_H^l[z]^{-1} \triangleq E_{z_H}^l[z]^{-1}\,E_{d_H}^l[z]^{-1}$, where $E_{z_H}^l[z] \in \mathbb{C}^{n\times n}$ and $E_{d_H}^l[z] \in \mathbb{C}^{n\times n}$ are given by (see [6] for details)

$$E_{z_H}^l[z]^{-1} \triangleq \prod_{k=1}^{n_c}\left\{\frac{1-c_k}{1-\overline{c_k}}\,\frac{1-z\,\overline{c_k}}{z-c_k}\,\eta_k\eta_k^H + U_kU_k^H\right\}, \quad (1)$$

$$E_{d_H}^l[z]^{-1} \triangleq \prod_{k=1}^{n_z}\left\{z\,\eta_{\infty_k}\eta_{\infty_k}^H + U_{\infty_k}U_{\infty_k}^H\right\}, \quad (2)$$

where $\eta_k \in \mathbb{C}^n$ and $\eta_{\infty_j} \in \mathbb{C}^n$ are left unitary vectors associated to the k-th finite and j-th infinite NMP zero, respectively, while $U_k \in \mathbb{C}^{n\times(n-1)}$ and $U_{\infty_k} \in \mathbb{C}^{n\times(n-1)}$ are unitary matrices satisfying $\eta_k\,\eta_k^H + U_kU_k^H = \eta_{\infty_j}\,\eta_{\infty_j}^H + U_{\infty_j}\,U_{\infty_j}^H = I_n$.

The symbol $\delta[k]$ denotes the Kronecker delta, while $\mathscr{E}\{\cdot\}$ denotes the expectation operator. The 2-norm of a function $P[z]$, analytic for $|z| = 1$, is defined as usual [13], [3]. Under the inner product that induces the 2-norm, $\mathbb{R}\mathscr{H}_2$ and $\mathbb{R}\mathscr{H}_2^{\perp}$ are orthogonal sets. Thus,

$$\|\,P[z]\,\|_2^2 = \left\|\{P[z]\}_{\mathbb{R}\mathscr{H}_2^{\perp}}\right\|_2^2 + \left\|\{P[z]\}_{\mathbb{R}\mathscr{H}_2}\right\|_2^2, \quad (3)$$

where $\{P[z]\}_{\mathbb{R}\mathscr{H}_2^{\perp}} \in \mathbb{R}\mathscr{H}_2^{\perp}$, and $\{P[z]\}_{\mathbb{R}\mathscr{H}_2} \in \mathbb{R}\mathscr{H}_2$. Defining $\{P[z]\}_{\mathbb{R}\mathscr{H}_\infty}$ as the part of $P[z] \in \mathbb{R}\mathscr{H}_\infty$ and using (3), it is possible to show that (see appendix A in [14])

$$\begin{aligned}\left\|\{P[z]\}_{\mathbb{R}\mathscr{H}_2^{\perp}} + \{P[z]\}_{\mathbb{R}\mathscr{H}_\infty}\right\|_2^2 &= \\ &\left\|\{P[z]\}_{\mathbb{R}\mathscr{H}_2^{\perp}} - \{P[z]\}_{\mathbb{R}\mathscr{H}_2^{\perp}}\Big|_{z=0}\right\|_2^2 \\ &+ \left\|\{P[z]\}_{\mathbb{R}\mathscr{H}_2^{\perp}}\Big|_{z=0} + \{P[z]\}_{\mathbb{R}\mathscr{H}_\infty}\right\|_2^2. \quad (4)\end{aligned}$$

III. OPTIMAL REGULATION PERFORMANCE FOR TALL SYSTEMS

A. Problem Formulation

In this section, we consider the closed loop configuration of Figure 1, where $G[z] \in \mathbb{C}^{n\times m}$ is a MIMO plant, $n \geq m$,

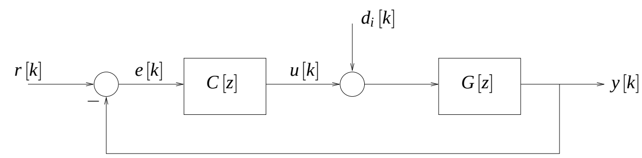

Fig. 1. One degree of freedom feedback loop.

and $C[z] \in \mathbb{C}^{m\times n}$ is the LTI controller. The signals $u[k] \in \mathbb{R}^m$, $r[k] \in \mathbb{R}^n$, $e[k] \in \mathbb{R}^n$, $d_i[k] \in \mathbb{R}^m$ and $y[k] \in \mathbb{R}^n$ are the control input, the reference, the tracking error, the input disturbance, and the system response, respectively. For the system in Figure 1, we measure the closed loop performance by using

$$J = \sum_{k=0}^{\infty} e[k]^T\,e[k]\,, \quad (5)$$

when $d_i[k] = \nu\,\delta[k]$, and where $\nu \in \mathbb{R}^m$. If the closed loop is internally stable, then

$$J = \|E[z]\|_2^2\,,$$

where $E[z] = -\left(I + G[z]\,C[z]\right)^{-1} G[z]\,\nu$ is the $\mathscr{Z}$-transform of the tracking error $e[k]$. Thus,

$$J = \left\|\left(I + G[z]\,C[z]\right)^{-1} G[z]\,\nu\right\|_2^2. \quad (6)$$

Since $G[z]$ is a rational matrix transfer function, it admits a doubly coprime factorization given by

$$G[z] \triangleq M_l[z]^{-1}\,N_l[z] = N_r[z]\,M_r[z]^{-1}\,,$$

where $N_l[z]$, $M_l[z]$, $N_r[z]$, $M_r[z] \in \mathbb{R}\mathscr{H}_\infty$ satisfy

$$\begin{bmatrix} X_l[z] & -Y_l[z] \\ -N_l[z] & M_l[z]\end{bmatrix}\begin{bmatrix} M_r[z] & Y_r[z] \\ N_r[z] & X_r[z]\end{bmatrix} = I\,, \quad (7)$$

for some $X_l[z]$, $Y_l[z]$, $X_r[z]$, $Y_r[z] \in \mathbb{R}\mathscr{H}_\infty$. All controllers which internally stabilize $G[z]$ can be written as [15]

$$\begin{aligned} C[z] &\triangleq \left(Y_r[z] - M_r[z]\,Q[z]\right)\left(N_r[z]\,Q[z] - X_r[z]\right)^{-1} \\ &= \left(Q[z]\,N_l[z] - X_l[z]\right)^{-1}\left(Y_l[z] - Q[z]\,M_l[z]\right), \quad (8)\end{aligned}$$

where $Q[z] \in \{\mathbb{C}^{m\times n} \cap \mathbb{R}\mathscr{H}_\infty\}$. Using (8) and (7) in (6), we have that

$$J = \left\|\left(X_r[z]\,N_l[z] - N_r[z]\,Q[z]\,N_l[z]\right)\nu\right\|_2^2. \quad (9)$$

To simplify the analysis, we assume that ν is a random vector satisfying $\mathscr{E}\{\nu\} = 0$ and $\mathscr{E}\left\{\nu\nu^T\right\} = I_m$. Applying the expectation operator to (9), and considering our assumptions on ν, we have that

$$\mathscr{E}\{J\} = \|X_r[z]\,N_l[z] - N_r[z]\,Q[z]\,N_l[z]\|_2^2\,. \quad (10)$$

The aim of this paper is minimizing the functional given in (10), subject to $Q[z] \in \mathbb{R}\mathscr{H}_\infty$ for several plant types, i.e., we aim at finding

$$J_1^{\text{opt}} \triangleq \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty} \|X_r[z]\,N_l[z] - N_r[z]\,Q[z]\,N_l[z]\|_2^2\,. \quad (11)$$

B. Optimal Regulation Performance: SIMO Case

The next theorem presents a closed form expression for J_1^{opt} in the SIMO plant case:

Theorem 1: Consider the problem in (11), a plant $G[z] = G_A[z] \in \mathbb{C}^{n\times 1}$, with non repeated finite NMP zeros c_i ($i = 1,\ldots,n_c$), and n_z NMP zeros at infinity. Furthermore, suppose that $G_A[z]$ has no zeros on $|z|=1$. Define $\hat{N}_l[z] \triangleq N_l[z]\,E^r_{N_l}[z]^{-1}$, and a power series expansion for $\Theta_{N_r}[z]\,X_l[z]$ as

$$\Theta_{N_r}[z]\,X_l[z] \triangleq \sum_{k=0}^{\infty} \alpha_i z^{-i},$$

where $\alpha_i \in \mathbb{R}$. Then, the optimal value of the functional given by (11) is achieved with $Q[z] = Q_t^{\text{opt}}[z]$, where

$$\begin{aligned} Q_t^{\text{opt}}[z] &\triangleq \arg\min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty} \mathscr{E}\{J\} \\ &= \Theta_{N_r}[z]^{-1}\left(R_{12}[0]+K_2[z]\right)\hat{N}_l[z]^{\dagger}, \end{aligned}$$

$K_2[z] \in \mathbb{R}\mathscr{H}_\infty$ is given by

$$\begin{aligned} K_2[z] \triangleq{}& \prod_{i=1}^{n_c}\frac{1-z\overline{c_i}}{z-c_i}\sum_{i=n_z}^{\infty}\alpha_i z^{n_z-i} \\ &- \sum_{j=1}^{n_c}\left\{m_j\,\frac{1-z\overline{c_j}}{z-c_j}\sum_{i=n_z}^{\infty}\alpha_i c_j^{n_z-i}\right\}, \end{aligned}$$

and

$$\begin{aligned} R_{12}[0] &\triangleq -\sum_{j=1}^{n_c}\left\{m_j\sum_{i=n_z}^{\infty}\alpha_i c_j^{n_z-i-1}\right\}, \\ m_j &\triangleq \prod_{\substack{k=1\\k\neq j}}^{n_c}\frac{1-c_j\overline{c_k}}{c_j-c_k}, \end{aligned}$$

and $\hat{N}_l[z]^{\dagger} \in \mathbb{R}\mathscr{H}_\infty$ is any left inverse of $\hat{N}_l[z]$. On the other hand, the optimal performance is given by

$$\begin{aligned} J_1^{\text{opt}} = \sum_{i=0}^{n_z-1}\alpha_i^2 + \sum_{h=1}^{n_c}\sum_{l=1}^{n_c}\Bigg\{ m_h\,m_l\,\frac{\left(|c_h|^2-1\right)\left(|c_l|^2-1\right)}{c_h\,c_l-1} \\ \times \sum_{i=n_z}^{\infty}\alpha_i c_h^{n_z-i-1}\sum_{j=n_z}^{\infty}\alpha_i c_l^{n_z-j-1}\Bigg\}, \end{aligned}$$

where all symbols are as above.

Proof: We have from (7), that $X_r[z]\,N_l[z] = N_r[z]\,X_l[z]$. Then, pre-multiplying the expression (11) by the unitary factor

$$U[z] = \begin{bmatrix} \Phi_{N_r}[z]^{\sim} \\ I - \Phi_{N_r}[z]\,\Phi_{N_r}[z]^{\sim} \end{bmatrix},$$

we obtain

$$J_1^{opt} = \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty} \left\|\Theta_{N_r}[z]\,X_l[z] - \Theta_{N_r}[z]\,Q[z]\,N_l[z]\right\|_2^2.$$

If we next make use of the right unitary factor $E^r_{N_l}[z]$, then

$$J_1^{opt} = \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|\Theta_{N_r}[z]\,X_l[z]\,E^r_{N_l}[z]^{-1} - \Theta_{N_r}[z]\,Q[z]\,\hat{N}_l[z]\right\|_2^2. \quad (12)$$

Since, for SIMO systems,

$$E^r_{N_l}[z] \triangleq \frac{1}{z^{n_z}}\prod_{i=1}^{n_c}\frac{z-c_i}{1-z\overline{c_i}},$$

we have that (12) reduces to

$$J_1^{opt} = \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|W[z]+K[z]-\Theta_{N_r}[z]\,Q[z]\,\hat{N}_l[z]\right\|_2^2, \quad (13)$$

where

$$\begin{aligned} W[z] &\triangleq \prod_{i=1}^{n_c}\frac{1-z\overline{c_i}}{z-c_i}\sum_{i=0}^{n_z-1}\alpha_i z^{n_z-i}, \\ K[z] &\triangleq \prod_{i=1}^{n_c}\frac{1-z\overline{c_i}}{z-c_i}\sum_{i=n_z}^{\infty}\alpha_i z^{n_z-i}. \end{aligned}$$

Now, we observe that $W[z]$ is analytic for $|z|<1$. Therefore, $W[z] \in \mathbb{R}\mathscr{H}_2^{\perp}$. However, $K[z]$ has stable and unstable poles. Using the partial fraction described in [16] for expanding $K[z]$, we have that

$$K[z] \triangleq \underbrace{\sum_{j=1}^{n_c}\left\{m_j\,\frac{1-z\overline{c_j}}{z-c_j}\sum_{i=n_z}^{\infty}\alpha_i c_j^{n_z-i}\right\}}_{R_{12}[z]} + K_2[z],$$

where $K_2[z] \in \mathbb{R}\mathscr{H}_\infty$, and $R_{12}[z]$ has only unstable terms. Therefore, $R_{12}[z] \in \mathbb{R}\mathscr{H}_2^{\perp}$. Since $\left(W[z]+R_{12}[z]\right) \in \mathbb{R}\mathscr{H}_2^{\perp}$ and $\left(K_2[z]-\Theta_{N_r}[z]\,Q[z]\,\hat{N}_l[z]\right) \in \mathbb{R}\mathscr{H}_\infty$, we can use (4), to write (13) as

$$\begin{aligned} J_1^{opt} ={}& \|W[z]+R_{12}[z]-W[0]-R_{12}[0]\|_2^2 \\ &+ \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|W[0]+R_{12}[0]+K_2[z]-\Theta_{N_r}[z]\,Q[z]\,\hat{N}_l[z]\right\|_2^2. \end{aligned} \quad (14)$$

Since that $W[0]=0$ and the second term on the RHS of (14) can be made zero choosing $Q[z] = Q_t^{\text{opt}}[z] \in \mathbb{R}\mathscr{H}_\infty$, we conclude that

$$J_1^{\text{opt}} = \|W[z]+R_{12}[z]-R_{12}[0]\|_2^2. \quad (15)$$

Applying the definition of the 2-norm, (15) reduces to

$$\begin{aligned} J_1^{\text{opt}} = \frac{1}{2\pi j}\oint &\{W[z]+R_{12}[z]-R_{12}[0]\}^{\sim} \\ &\times\{W[z]+R_{12}[z]-R_{12}[0]\}\,\frac{dz}{z}, \end{aligned}$$

where integral is over the unit circle, traveled counterclockwise. Using Lema 1 in [17], it is straightforward to prove that $J_1^{opt} = \gamma_1+\gamma_2-\gamma_3$, where

$$\begin{aligned} \gamma_1 &\triangleq \frac{1}{2\pi j}\oint W[z]^{\sim}\,W[z]\,\frac{dz}{z}, \\ \gamma_2 &\triangleq \frac{1}{2\pi j}\oint R_{12}[z]^{\sim}\,R_{12}[z]\,\frac{dz}{z}, \\ \gamma_3 &\triangleq \frac{1}{2\pi j}\oint R_{12}[z]^{\sim}\,R_{12}[0]\,\frac{dz}{z}. \end{aligned}$$

It is possible to show that

$$\gamma_1 = \sum_{i=0}^{n_z-1}\alpha_i^2,$$

and that

$$\gamma_2 - \gamma_3 = \sum_{h=1}^{n_c}\sum_{l=1}^{n_c}\left\{ m_h\, m_l\, \frac{\left(|c_h|^2-1\right)\left(|c_l|^2-1\right)}{c_h\, c_l - 1} \times \sum_{i=n_z}^{\infty} \alpha_i\, c_h^{n_z-i-1} \sum_{j=n_z}^{\infty} \alpha_i\, c_l^{n_z-j-1} \right\}.$$

The proof is thus completed. ■

Theorem 1 gives a closed form expression for the optimal regulation performance for SIMO systems that have non repeated finite NMP zeros, but are otherwise arbitrary (cf. [11]).

Our result are much simplified if one imposes additional constraints on the plant. For instance, the next immediate corollary of Theorem 1 presents the best achievable regulation performance when the plant has one zero at infinity:

Corollary 1: Consider the setup, notation, and assumptions of Theorem 1. If $n_z = 1$, then

$$\begin{aligned} J_1^{\text{opt}} &= \left(\Theta_{N_r}[\infty]\, M_r[\infty]^{-1}\right)^2 \\ &\quad + \sum_{h=1}^{n_c}\sum_{l=1}^{n_c}\left\{ m_h\, m_l\, \frac{\left(|c_h|^2-1\right)\left(|c_l|^2-1\right)}{c_h\, c_l - 1} \right. \\ &\quad \times \left(\Theta_{N_r}[\infty]\, M_r[\infty]^{-1} - \Theta_{N_r}[c_h]\, M_r[c_h]^{-1}\right) \\ &\quad \left. \times \left(\Theta_{N_r}[\infty]\, M_r[\infty]^{-1} - \Theta_{N_r}[c_l]\, M_r[c_l]^{-1}\right)\right\}. \end{aligned}$$

■

The result of Corollary 1 is consistent with the results in [11], when particularized to the performance measure considered in this paper.

IV. OPTIMAL REGULATION PERFORMANCE FOR SQUARED-UP PLANTS

A. Problem Formulation

In this section, we consider that an additional control channel is added to a tall plant $G_A[z] \in \mathbb{C}^{n\times(n-1)}$. The resulting transfer function, say $G[z]$, is thus square. The structure of the augmented plant $G[z]$, as a function of the tall plant $G_A[z]$, is given by

$$G[z] \triangleq \left[\, G_A[z] \;\vdots\; G_n[z] \,\right], \tag{16}$$

where $G_n[z] \in \mathbb{C}^{n\times 1}$ is the column associated to the new (i.e., additional) control channel. As before, we consider the closed loop shown in Figure 1. We use the notation and definitions presented in the previous section. However, the dimensions of some signals and systems must be redefined: the control input $u[k] \in \mathbb{R}^n$, the input disturbance $d_i[k] \in \mathbb{R}^n$, the controller $C[z] \in \mathbb{C}^{n\times n}$, and the plant $G[z] \in \mathbb{C}^{n\times n}$.

As before, the performance is measured by (5). However, we will assume that only the original $n-1$ control channels are perturbed by $d_i[k]$, i.e., we assume that $v \in \mathbb{R}^n$ is such that $\mathscr{E}\{v\} = 0$ and

$$\mathscr{E}\left\{v\, v^T\right\} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

This assumption will allow us to fairly asses the benefits of augmenting $G_A[z]$ with an additional control channel.

Proceeding as before, and assuming both $G_A[z]$ and $G[z]$ to be stable, we can write the problem of interest in this section as that of finding

$$J_2^{\text{opt}} \triangleq \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty} \|(I - G[z]\, Q[z])\, G_A[z]\|_2^2 . \tag{17}$$

The solution to this problem is derived in the next subsection.

B. Optimal Regulation Performance: Augmented Plant Case

The next results presents a characterization of J_2^{opt}, as defined in (17):

Theorem 2: Consider the problem formulated in (17), a plant $G[z] = E_G^l[z]\, \hat{G}[z] \in \mathbb{C}^{n\times n}$, stable, with non repeated finite NMP zeros c_i $(i = 1,\ldots,n_c)$, and n_z NMP zeros at infinity. Furthermore, assume that the plant $G[z]$ has no zeros on $|z| = 1$, and it has a partition given by (16). Also define $\hat{G}_A[z] = G_A[z]\, E_{G_A}^r[z]^{-1} \in \mathbb{C}^{n\times(n-1)}$. Then, the optimal value of the functional (17) is obtained by choosing $Q[z] = Q_s^{\text{opt}}[z]$, where

$$\begin{aligned} Q_s^{\text{opt}}[z] &\triangleq \arg\min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty} \mathscr{E}\{J\} \\ &= \hat{G}[z]^{-1}\left(R_2[z] - \sum_{k=1}^{n_c}\frac{A_k}{c_k}\right)\hat{G}_A[z]^{\dagger}, \end{aligned}$$

where $\hat{G}_A[z]^{\dagger} \in \mathbb{R}\mathscr{H}_\infty$ is a left inverse of $\hat{G}_A[z]$,

$$A_k \triangleq M_k \sum_{i=0}^{r}\sum_{j=i}^{\infty} C_i\, B_j\, c_k^{i-j},$$

$C_i \in \mathbb{R}^{n\times n}$ and $B_j \in \mathbb{R}^{n\times(n-1)}$ are the coefficients of the power series expansions

$$\hat{G}_A[z] \triangleq \sum_{j=0}^{\infty} B_j\, z^{-j},\; E_{d_G}^l[z]^{-1} \triangleq \sum_{i=0}^{r} C_i\, z^i, \tag{18}$$

$R_2[z] \in \mathbb{R}\mathscr{H}_\infty$ is given by

$$R_2[z] \triangleq E_{z_G}^l[z]^{-1}\sum_{i=0}^{r}\sum_{j=i}^{\infty} C_i\, B_j\, z^{j-i} - \sum_{k=1}^{n_c}\frac{A_k}{z - c_k},$$

with $E_{z_G}^l[z]^{-1}$ defined as in (1), and

$$M_k \triangleq L_{k,1,k-1}\left\{\frac{1-c_k}{1-\overline{c_k}}\left(1-|c_k|^2\right)\eta_k\,\eta_k^H\right\} L_{k,k+1,n_c},$$

with

$$L_{k,j,h} = \prod_{i=j}^{h}\left\{\frac{1-c_i}{1-\overline{c_i}}\,\frac{1-c_k\,\overline{c_i}}{c_k - c_i}\,\eta_i\,\eta_i^H + U_i\, U_i^H\right\}.$$

Moreover, the optimal value J_2^{opt} is given by

$$\begin{aligned} J_2^{\text{opt}} = \operatorname{trace}\Bigg\{ & T_1 + \sum_{l=1}^{n_c}\sum_{k=1}^{n_c}\frac{A_l^H A_k}{\overline{c_l}\, c_k\,(\overline{c_l}\, c_k - 1)} \\ & - \sum_{k=1}^{n_c}\frac{A_k^H}{\overline{c_k}}\, E_{z_G}^l\left[\overline{c_k}^{-1}\right]^{-1}\sum_{i=0}^{r}\sum_{j=0}^{i-1} C_i\, B_j\, \overline{c_k}^{\,j-i}\Bigg\}, \end{aligned}$$

where

$$T_1 \triangleq \frac{1}{2\pi j}\oint\left(\sum_{i=0}^{r}\sum_{j=0}^{i-1} B_j^T C_i^T z^{j-i} \times \sum_{i=0}^{r}\sum_{j=0}^{i-1} C_i B_j z^{i-j}\right)\frac{dz}{z}, \tag{19}$$

and the integral is over the unit circle, counterclockwise oriented.

Proof: By definition of $E_G^l[z]^{-1}$ and $E_{G_A}^r[z]^{-1}$,

$$J_2^{opt} = \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|E_G^l[z]^{-1}\hat{G}_A[z] - \hat{G}[z]Q[z]\hat{G}_A[z]\right\|_2^2.$$

Writing $E_G^l[z]^{-1} = E_{z_G}^l[z]^{-1}E_{d_G}^l[z]^{-1}$ (see (1) and (2)), and using the power series expansion in (18), we have that

$$\begin{aligned} J_2^{opt} &= \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|E_G^l[z]^{-1}\hat{G}_A[z] - \hat{G}[z]Q[z]\hat{G}_A[z]\right\|_2^2 \\ &= \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|E_{z_G}^l[z]^{-1}\sum_{i=0}^{r}\sum_{j=0}^{\infty} C_i B_j z^{i-j} - \hat{G}[z]Q[z]\hat{G}_A[z]\right\|_2^2 \\ &= \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|\underbrace{E_{z_G}^l[z]^{-1}\sum_{i=0}^{r}\sum_{j=0}^{i-1} C_i B_j z^{i-j}}_{O[z]} + \underbrace{E_{z_G}^l[z]^{-1}\sum_{i=0}^{r}\sum_{j=i}^{\infty} C_i B_j z^{i-j}}_{V[z]} - \hat{G}[z]Q[z]\hat{G}_A[z]\right\|_2^2. \end{aligned} \tag{20}$$

The term $O[z]$ contains only improper and unstable terms. Therefore $O[z]\in\mathbb{R}\mathscr{H}_2^\perp$. However, the term $V[z]$ has stable and unstable poles. For this reason, it is necessary to expand $V[z]$ in partial fractions:

$$V[z] \triangleq \underbrace{\sum_{k=1}^{n_c}\frac{A_k}{z-c_k}}_{R_1[z]} + R_2[z], \tag{21}$$

where $R_2[z]\in\mathbb{R}\mathscr{H}_\infty$ and $A_k \triangleq \lim_{z\to c_k}(z-c_k)V[z]$. Substituting (21) in (20), we have that

$$J_2^{opt} = \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|O[z]+R_1[z]+R_2[z]-\hat{G}[z]Q[z]\hat{G}_A[z]\right\|_2^2. \tag{22}$$

Since $(O[z]+R_1[z]) \in \mathbb{R}\mathscr{H}_2^\perp$ and $(R_2[z]-\hat{G}[z]Q[z]\hat{G}_A[z])\in\mathbb{R}\mathscr{H}_\infty$, and noting that $O[0]=0$, it is possible to use (4) into (22) to write

$$\begin{aligned} J_2^{opt} &= \|O[z]+R_1[z]-R_1[0]\|_2^2 \\ &\quad + \min_{Q[z]\in\mathbb{R}\mathscr{H}_\infty}\left\|R_1[0]+R_2[z]-\hat{G}[z]Q[z]\hat{G}_A[z]\right\|_2^2. \end{aligned} \tag{23}$$

The second term of the RHS in (23) can be made zero by choosing $Q[z]=Q_s^{\rm opt}[z]$. Therefore, the optimal value $J_2^{\rm opt}$ is given by

$$J_2^{\rm opt} = \|O[z]+R_1[z]-R_1[0]\|_2^2. \tag{24}$$

Applying the 2-norm definition to (24) we obtain

$$\begin{aligned} J_2^{\rm opt} = \operatorname{trace}\Big\{\frac{1}{2\pi j}\oint &\{O[z]+R_1[z]-R_1[0]\}^\sim \\ &\times\{O[z]+R_1[z]-R_1[0]\}\frac{dz}{z}\Big\}, \end{aligned}$$

where the integral is over the unit circle, traveled counterclockwise. In spite of the large number of cross products, $J_2^{opt} = \beta_1+\beta_2+\beta_3$, where

$$\begin{aligned} \beta_1 &\triangleq \operatorname{trace}\left\{\frac{1}{2\pi j}\oint O[z]^\sim O[z]\frac{dz}{z}\right\}, \\ \beta_2 &\triangleq \operatorname{trace}\left\{\frac{1}{2\pi j}\oint R_1[z]^\sim O[z]\frac{dz}{z}\right\}, \\ \beta_3 &\triangleq \operatorname{trace}\left\{\frac{1}{2\pi j}\oint R_1[z]^\sim\{R_1[z]-R_1[0]\}\frac{dz}{z}\right\}. \end{aligned}$$

It is straightforward to show (applying the techniques used in [17]) that the above integrals become (see (19))

$$\begin{aligned} \beta_1 &= \operatorname{trace}\{T_1\}, \\ \beta_2 &= \operatorname{trace}\left\{-\sum_{k=1}^{n_c}\frac{A_k^H}{\overline{c_k}}E_{z_G}^l\left[\overline{c_k}^{-1}\right]^{-1}\sum_{i=0}^{r}\sum_{j=0}^{i-1}C_iB_j\overline{c_k}^{j-i}\right\}, \\ \beta_3 &= \operatorname{trace}\left\{\sum_{l=1}^{n_c}\sum_{k=1}^{n_c}\frac{A_l^H A_k}{\overline{c_l}c_k(\overline{c_l}c_k-1)}\right\}. \end{aligned}$$

The result thus follows. ■

Remark 1: It is possible to extend Theorem 2 to more general situations, where more than one additional control channel is added. This extension can be obtained by changing the assumption on the random disturbance vector covariance matrix to

$$\mathscr{E}\left\{vv^T\right\} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0\end{bmatrix},$$

where k is the number of additional control channels. With this change, it is possible to develop a closed form for $J_2^{\rm opt}$ by proceeding as in the proof of Theorem 2. ■

The result derived in this section shows that the presence of NMP zeros and delays in the system worsen the regulation performance of the augmented plant. However, as intuition suggests, the performance achieved in that case is always better than (or at least equal to) the one obtained when controlling the original tall plant:

Corollary 2: Consider the setup and assumptions of Sections III and IV. If a stable plant $G_A[z]\in\mathbb{C}^{n\times(n-1)}$ is augmented to a stable plant $G[z]=\left[\,G_A[z] \;\vdots\; G_n[z]\,\right]\in\mathbb{C}^{n\times n}$, then $J_2^{\rm opt}\le J_1^{\rm opt}$.

Proof: It is straightforward to prove the inequality by observing that there exists $Q[z]\in\mathbb{C}^{n\times n}$ such that

$$Q[z] = \begin{bmatrix} Q_t^{\rm opt}[z] \\ \hdashline 0 \end{bmatrix}.$$

Therefore, the optimal value $J_2^{\rm opt}$ is always less than or equal to $J_1^{\rm opt}$, which ends the proof. ■

It would be useful to develop a closed form expression for the difference $\Delta J \triangleq J_2^{\rm opt} - J_1^{\rm opt}$. We leave that for future research. A numerical illustration of the benefits of adding additional control channels is presented below.

V. EXAMPLE

Consider a SITO system defined as

$$G_A[z] = \left[\frac{3\,(z-c)}{z^2\,(z-0.8)} \quad \frac{2\,(z-c)}{z^2\,(z-0.2)} \right]^T ,$$

where $c > 1$. The plant $G_A[z]$ has a NMP zero located at $z = c$, and relative degree $n_z = 2$. In order to improve achievable performance, we propose to augment the system to

$$G[z] = \begin{bmatrix} \frac{3\,(z-c)}{z^2\,(z-0.8)} & \frac{(z-0.3)}{z^2} \\ \frac{2\,(z-c)}{z^2\,(z-0.2)} & \frac{2\,(z-0.3)}{z\,(z-0.2)} \end{bmatrix} .$$

The augmented plant $G[z]$ has a only one finite NMP zero located at $z = c$, three NMP zeros at infinity, and two additional MP zeros located at $z = 0.3$ and $z = -0.4$.

Under the conditions described above, the optimal value J_1^{opt} is given by

$$J_1^{\text{opt}} = \sum_{i=0}^{1} \alpha_i^2 + (c^2-1)\left\{ \sum_{i=2}^{\infty} \alpha_i c^{1-i} \right\}^2 ,$$

where α_i is defined as in Theorem 1. On the other hand, the optimal value J_2^{opt} is given by

$$J_2^{\text{opt}} = \frac{A_1^H A_1}{c^2\,(c^2-1)} + \frac{1}{2\pi j} \oint \left(\sum_{i=0}^{2} \sum_{j=0}^{i-1} B_j^T C_i^T z^{j-i} \times \sum_{i=0}^{2} \sum_{j=0}^{i-1} C_i B_j z^{i-j} \right) \frac{dz}{z} ,$$

where A_1, B_j, and C_i are defined as in Theorem 2. Figure 2 shows the ratio $R \triangleq (J_1^{\text{opt}} - J_2^{\text{opt}})(J_2^{\text{opt}})^{-1}$ as a function of c. Although the considered example is very simple, we see that the dependence of R on c is non-trivial. Consistent with Corollary 2, Figure 2 shows that $R \geq 0$.

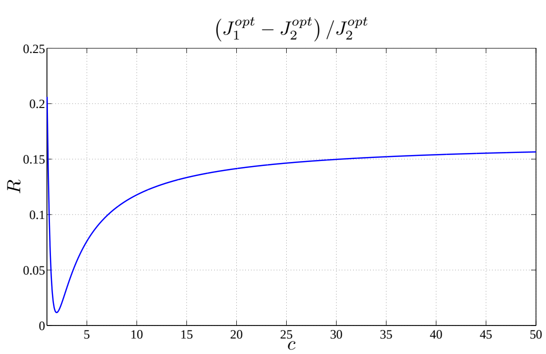

Fig. 2. Improving ratio R as a function of NMP zero c.

This example illustrates briefly the two results presented in this article: the optimal regulation performance for a plant $G_A[z] \in \mathbb{C}^{n\times(n-1)}$, and the same index in its augmented structure $G[z] \in \mathbb{C}^{n\times n}$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we have developed a closed form expression for the best achievable regulation performance for SIMO systems. This expression is a function of the NMP zeros, and the plant structure. Our results apply to any SIMO plant with non repeated finite NMP zeros. We have also derived an expression for the best regulation performance for tall plants that can be squared-up by the addition of one additional control channel. Unsurprisingly, our result shows that the addition of additional control inputs is always beneficial for closed loop performance.

Future work on the subject should consider tall plants of arbitrary dimensions, and situations where an arbitrary number of additional control inputs is added.

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