Cohomological Invariants of algebraic curves

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Some notation

we fix a base field k_0 and a prime number p. We will always assume that the characteristic of k_0 is different from p, and that we have a fixed primitive p-th root of unit ζ in k_0.
If X is a k_0-scheme we will denote by $\mathrm{H}^i(X)$ the étale cohomology ring of X with coefficients in $\mathbb{Z}/p\mathbb{Z}$. If R is a k_0-algebra, we set $\mathrm{H}^\bullet(R) = \mathrm{H}^\bullet(\mathrm{Spec}(R))$.
All schemes and algebraic stacks considered will be of finite type over k_0 and quasi-separated.

Classical vs new

A näive definition of cohomological invariants for algebraic stacks:

- Given an algebraic stack $\mathscr{M}$, let $P_{\mathscr{M}}$ be the functor of isomorphism classes of maps $\mathrm{Spec}(K) \to \mathscr{M}$
- A cohomological invariant for $\mathscr{M}$ is a natural transformation

$$P_{\mathscr{M}} \to \mathrm{H}^{\bullet}$$

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This definition is incomplete. In fact, it does not even distinguish between a scheme and the disjoint union of its points.
To solve this problem, we introduce a *continuity condition*.

Continuity condition

We restrict to natural transformations satisfying a technical condition, which can roughly be stated as:

Let R be a DVR and $f : \mathrm{Spec}(R) \to \mathscr{M}$ a map. The value of a cohomological invariant on the closed point of $\mathrm{Spec}(R)$ is determined by its value at the generic point.

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We write $\mathrm{Inv}^\bullet(\mathscr{M})$ for the ring of natural transformations $P_{\mathscr{M}} \to \mathrm{H}^\bullet$ satisfying the continuity condition.
There is a natural map sending étale cohomology with coefficients in $\mathbb{Z}/p\mathbb{Z}$ to cohomological invariants. In general it is neither surjective nor injective.

Choice of topology

Cohomological invariants have an obvious pullback map induced by composition. We want to find the right Grothendieck topology to make it into a sheaf.

- The étale and smooth topologies are too fine: pulling back cohomological invariants through an étale covering is in general not injective.
- The Zariski topology is too coarse: we want algebraic stacks to be covered by schemes in our topology.

We need to look for a compromise between these options.

Lifting points

Definition

We say that a representable map of algebraic stacks $f : \mathscr{M} \to \mathscr{N}$ has the lifting property if for every map $p : \mathrm{Spec}(K) \to \mathscr{N}$ there is a lifting

$$\begin{array}{ccc} & & \mathscr{M} \\ & \overset{p'}{\nearrow} & \downarrow f \\ \mathrm{Spec}(K) & \xrightarrow{\quad p \quad} & \mathscr{N} \end{array}$$

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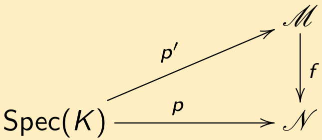

A *Nisnevich* (resp. *smooth-Nisnevich*) covering is a representable étale (resp. smooth) map having the lifting property.

Cohomolgical invariants are a sheaf in the *Nisnevich* and *smooth-Nisnevich* topologies. In general even Deligne-Mumford stacks will not be covered by schemes in the *Nisnevich* topology, so we restrict to the latter.

A complete description

Theorem

Consider the functor $H^\bullet_{ét}(-, \mathbb{Z}/p\mathbb{Z})$ sending a smooth algebraic stack to its étale cohomology. There is a natural map

$$H^\bullet_{ét}(-, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{j} \mathrm{Inv}^\bullet(-), \quad j(\alpha)(p) = p^*(\alpha)$$

for $\alpha \in H^\bullet_{ét}(\mathscr{M}, \mathbb{Z}/p\mathbb{Z})$ and $p : \mathrm{Spec}(K) \to \mathscr{M}$.
This map extends to a map

$$(H^\bullet_{ét}(-, \mathbb{Z}/p\mathbb{Z}))^{sm\text{-}Nis} \xrightarrow{\tilde{j}} \mathrm{Inv}^\bullet$$

where $(-)^{sm\text{-}Nis}$ denotes the smooth-Nisnevich sheafification.
The map $\tilde{j}$ is an isomorphism.

Idea of the proof

- On schemes we prove that a cohomological invariant only depends on its value at the generic point. The ring of possible values must satisfy some ramification conditions, and it is know as the *unramified cohomology* ring.

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- The unramified cohomology of a smooth scheme is classically known to be isomorphic to the Zariski sheafification of étale cohomology due to the Bloch-Ogus theorem. The latter maps to $\mathrm{Inv}^\bullet$ through the map $\tilde{j}$, obtaining the isomorphism on schemes.

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- We can use to sheaf condition to infer the general result from the result on schemes.

Invariance results

We can use the explicit description on schemes to infer the following:

Corollary

- *Let* $\mathscr{E} \to \mathscr{M}$ *be a vector bundle. Then the pullback* $\mathrm{Inv}^\bullet(\mathscr{M}) \to \mathrm{Inv}^\bullet(\mathscr{E})$ *is an isomorphism.*
- *Let* $\mathscr{N}$ *be a closed substack of codimension 2 or more. Then the pullback* $\mathrm{Inv}^\bullet(\mathscr{M}) \to \mathrm{Inv}^\bullet(\mathscr{M} \smallsetminus \mathscr{N})$ *is an isomorphism.*

A classical application

with the two corollaries we easily obtain a new proof of this strong classical result by B.Totaro:

Theorem (Totaro)

Let G be an affine algebraic group smooth over k_0. Suppose that we have a representation V of G and a closed subset $Z \subset V$ such that the codimension of Z in V is 2 or more, and the complement $U = V \setminus Z$ is a G-torsor. Then the group of cohomological invariants of G is isomorphic to the unramified cohomology of U/G.

The tool

We want to compute some nontrivial ring of cohomological invariants. Our main tool will be the *Chow ring with coefficients*, introduced by M.Rost. Given a smooth scheme X it is a bigraded ring $A^{\bullet,\bullet}(X)$.
If we consider the ring $A^{\bullet,0}(X)$ we obtain the usual Chow ring tensored by $\mathbb{Z}/p\mathbb{Z}$. If we consider the ring $A^{0,\bullet}(X)$ we get the unramified cohomology of X.

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We aim to understand the ring $A^{0,\bullet}(X)$ for some smooth-Nisnevich cover of the stack $\mathscr{M}$ we're interested in, and then check the gluing conditions.
Even better, for quotient stacks $[X/G]$ we have an equivariant version $A_G^{\bullet,\bullet}(X)$ of the theory that allows us to skip checking the gluing conditions altogether. It was introduced by B.Totaro and P.Guillot.

The main result

Theorem

Suppose our base field k_0 is algebraically closed, of characteristic different from 2, 3. Let $\mathscr{H}_g$ be the stack of hyperelliptic curves of genus g.

- *Suppose g is even. For $p = 2$ a basis for $\mathrm{Inv}^\bullet(\mathscr{H}_g)$ as a graded $\mathbb{F}_2$-module is $\{1, x_1, \ldots, x_{g+2}\}$, where the degree of x_i is i.*
  If $p \neq 2$, a basis for $\mathrm{Inv}^\bullet(\mathscr{H}_g)$ is $\{1, x_1\}$ if $2g + 1$ is divisible by p, and $\{1\}$ otherwise.
- *For $p = 2$ a basis for $\mathrm{Inv}^\bullet(\mathscr{H}_3)$ as a graded $\mathbb{F}_2$-module is $\{1, x_1, x_2, w_2, x_3, x_4, x_5\}$, where the degree of x_i is i and w_2 comes from the cohomological invariants of PGL_2.*
  If $p \neq 2$, then the cohomological invariants of $\mathscr{H}_3$ are trivial for $p \neq 7$ and freely generated by 1 and x_1 for $p = 7$.

A presentation for $\mathscr{H}_g$

We use a very explicit description of the stacks of hyperelliptic curves, by Arsie and Vistoli.

Theorem (A.Arsie, A.Vistoli)

Consider the affine space $\mathbb{A}^{2g+3}$, seen as the space of all binary forms $\phi(x) = \phi(x_0, x_1)$ of degree $2g+2$. Denote by X_g the open subset consisting of nonzero forms with distinct roots. Consider the action of GL_2 on X_g defined by $A(\phi(x)) = \det(A)^g \phi(A^{-1}x)$. For an even g we have

$$\mathscr{H}_g \simeq [X_g/GL_2]$$

If g is odd, let $PGL_2 \times G_m$ act on X_g by $([A], \alpha)(f)(x) = \mathrm{Det}(A)^{g+1}\alpha^{-2}f(A^{-1}(x))$. We have

$$\mathscr{H}_g = [X_g/(PGL_2 \times G_m)]$$

Stratifying the problem

We want to understand the ring $A_G^{0,\bullet}(X_g)$, where G is respectively GL_2 for even g and PGL_2 for odd g.
We use a variant of the *stratification method*, first used by G.Vezzosi in his phd thesis to compute the Chow ring of $BPGL_3$, and by P.Guillot to compute cohomological invariants of algebraic groups.
Given a representation V of an algebraic group G we find some closed subset Z such that good things (e.g. being able to reduce to a simpler group) happen for both $V \setminus Z$ and Z, compute the Chow rings of both, and then use the localization sequence to get the result.

The stratification

In our case we are already working with an open subset of a representation, namely the space of nondegenerate binary forms of degree $2g+2$. We need to get enough information on the equivariant Chow Groups with coefficients of the closed subset Δ consisting of degnerate forms.

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First we take the quotient by the multiplicative group $\mathbb{G}_m$ so that we are working with subschemes of $\mathbb{P}^{2g+2}$. Our stratification is given by

$$\mathbb{P}^{2g+2} \supset \Delta_{1,2g+2} \supset \Delta_{2,2g+2} \supset \ldots \supset \Delta_{g+1,2g+2}$$

The closed subscheme $\Delta i, r$ of $\mathbb{P}^r$ is composed of those forms of degree r that are divisible by the square of a form of degree i.

Idea of the proof

The proof of the main theorem is done by induction starting from the following two lemmas:

Proposition

Let $\pi_{r,i} : \mathbb{P}^{r-2i} \times \mathbb{P}^{i} \to \Delta_{i,r}$ *be the map induced by* $(f, g) \to fg^2$*. The equivariant morphism* $\pi_{i,r}$ *restricts to a universal homeomorphism on* $\Delta_{i,r} \smallsetminus \Delta_{i+1,r}$*.*
Moreover, the inverse image of $\Delta_{i+1,r}$ *is* $\Delta_{1,r-2i} \times \mathbb{P}^{i}$*.*

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Moreover, the inverse image of $\Delta_{i+1,r}$ is $\Delta_{1,r-2i} \times \mathbb{P}^i$.

Proposition

A universal homeomorphism induces an isomorphism on Chow groups with coefficients.

Idea of the proof - 2

The two lemmas show that the chow groups with coefficients of $\Delta_{i,r} \smallsetminus \Delta_{i,r+1}$ are isomorphic to those of $(\mathbb{P}^{r-2i} \smallsetminus \Delta_{1,r-2i}) \times \mathbb{P}^i$.
We have a formula for the chow groups with coefficients of a projective bundle, so we have reduced the computation of $A_G^{\bullet,\bullet}(\Delta_{i,r})$ to something concerning $A_G^{\bullet,\bullet}(\Delta_{1,r-2i})$ and $A_G^{\bullet,\bullet}(\Delta_{i+1,r})$.

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We have a formula for the chow groups with coefficients of a projective bundle, so we have reduced the computation of $A^{\bullet,\bullet}_{G}(\Delta_{i,r})$ to something concerning $A^{\bullet,\bullet}_{G}(\Delta_{1,r-2i})$ and $A^{\bullet,\bullet}_{G}(\Delta_{i+1,r})$.
The index r can only be as small as 2, and the index i can only be as big as $r/2$. We start from the bottom case of $\Delta_{1,2}$, which is universally homeomorphic to $\mathbb{P}^1$, and using the reduction above we can inductively compute all the invariants we need to conclude.

Thoughts for the future

We still lack a way to understand the product structure of $\mathrm{Inv}^\bullet(\mathscr{H}_g)$ or to produce invariants for $\mathscr{M}_g, g \geq 3$. One idea is to try to reduce to classical cohomological invariants. Suppose our base field contains a q-th root of unit for a prime q.

Given a family of curves $\mathcal{C} \xrightarrow{f} X$ we can consider the sheaf $R_{f_*}(\mathbb{Z}/q\mathbb{Z})$ on X, or equivalently the q-torsion in the Jacobian of $\mathcal{C}$. It is a form of $\mathbb{Z}/q\mathbb{Z}^{2g}$ with a nondegenerate symplectic pairing, so it induces a map $\mathscr{M}_g \to BSp(2g, \mathbb{F}_q)$.

To the author's knowledge the cohomological invariants of $Sp(2g, \mathbb{F}_q)$ are not known. Hopefully computing them and studying the maps $\mathscr{M}_g \to BSp(2g, \mathbb{F}_q)$ can shed some light on the cohomological invariants of $\mathscr{M}_g$ and possibly be instrumental in creating some stable cohomological invariant classes.

Thank you for your attention!