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Cohomological invariants of algebraic curves

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0. INTRODUCTION

The study of algebraic curves has been a central part of algebraic geometry since its beginnings. From the Riemann-Roch theorem to to the introduction of the Jacobian, to the study of the moduli spaces of curves, algebraic curves have been a fruitful source of inspiration and new questions throughout the last two centuries.

One of the most prominent new concepts brought about by the study of algebraic curves is the concept of algebraic stacks. The categorical concept of stack is due to Alexander Grothendieck. It was first presented in Grothendieck's 1960 Seminaire Bourbaki [Gro60], and in Grothendieck's Séminaire de Gémoetrie Algébrique [Gro68]. It was later treated in Giraud's "Méthode de la descente" [Gir64].

The idea of algebraic stacks was first introduced by Mumford in "Picard groups of moduli problems" [Mum65], independently from Grothendieck and Giraud's work on descent theory. The concept was proven to be a successful one shortly after, when Deligne and Mumford used it to prove the irreducibility of the moduli space of smooth curves of a given genus in [DM69]. The concept of a Deligne-Mumford stack, which encompasses all of the stacks of stable algebraic curves, is named after them. Later, the more general concept of algebraic stack (also called "Artin stack") was introduced by Michael Artin in [Art73].

Algebraic stacks extend the idea of moduli spaces. Given a category $\mathcal{C}$ of objects that we want to study (e.g. the category of smooth genus g algebraic curves), we would like to have a space $M_{\mathcal{C}}$ such that elements of our category (e.g. flat morphisms $E \to X$, with fibers algebraic curves of genus g) correspond bijectively to morphisms to $M_{\mathcal{C}}$.

This is often impossible to realize as an ordinary space. As pointed out as early as 1959 by Grothendieck in a letter to Serre, a fundamental obstruction to the existence of a (fine) moduli space $M_{\mathcal{C}}$ is given by the existence of automorphisms in $\mathcal{C}$. This is solved by the notion of algebraic stacks, where rather than a space we consider a whole category $\mathscr{M}$, on which we can do geometry thanks to the existence of a "covering" $\underline{X} \to \mathscr{M}$, where X is a scheme with underlying category $\underline{X}$, and the morphism is in an appropriate sense smooth and surjective.

Algebraic stacks work as a framework for the study of families of geometric objects, and their geometric and arithmetic properties reflect the inner structure of the underlying category. This makes all the more important to be able to define and compute the basic invariants that are studied in algebraic geometry. Algebraic stacks have coherent sheaves, cohomology, Picard groups and intersection theory all of which have been widely studied.

In this thesis we consider a type of invariants which has, until now, been only treated for a very restricted class of algebraic stacks, the classifying stacks for smooth algebraic groups. In fact, the invariants we consider were more often then not treated as an invariant of the algebraic group G rather than an invariant of the classifying stack BG.

In the classical setting, the *cohomological invariants* of an algebraic group G are meant to be an arithmetic equivalent to the theory of characteristic classes in topology, which arose in the early second half of the twentieth century due to the work of Stiefel and Whitney, after whom the Stiefel-Whitney classes are named. Given a topological group G and a space X, a characteristic class is a way to associate to any principal G-bundle an element in the cohomology of X in a functorial way. In the same way, a cohomological invariant for an algebraic group G over a field k is a way to associate to any G-torsor over an extension L of k an element in the Galois cohomology $H^i_{Gal}(L, F)$ in a functorial way. Common choices for the Galois module F are the constant modules $\mathbb{Z}/p$ for a prime p and the Tate twists $\mathbb{Z}/p(j)$.

The aim of this thesis is to turn this concept on its head and consider the group of cohomological invariants as an invariant of the classifying stack BG rather than the group G, allowing us to extend the idea to general algebraic stacks.

I Cohomological invariants of algebraic groups

Some notation: we fix a base field k_0 and a prime number p. We will always assume that the characteristic of k_0 is different from p. If X is a k_0-scheme we will denote by $\mathrm{H}^i(X)$ the étale cohomology ring of X with coefficients in $\mu_p^{\otimes i}$ (here $\mu_p^{\otimes 0} := \mathbb{Z}/p\mathbb{Z}$), and by $\mathrm{H}^\bullet(X)$ the direct sum $\oplus_i \mathrm{H}^i(X)$. If R is a k_0-algebra, we set $\mathrm{H}^\bullet(R) = \mathrm{H}^\bullet(\mathrm{Spec}(R))$.

An early example of cohomological invariants dates back to Witt's seminal paper [Wit37], where the Hasse-Witt invariants of quadratic forms were defined. Many other invariants of quadratic forms, such as the Stiefel-Whitney classes and the Arason invariant were studied before the general notion of étale cohomological invariant was introduced.

This was inspired by the theory of characteristic classes in topology, and is naturally stated in functorial terms as follows.

Denote by (Field/k_0) the category of extensions of k_0. Its objects are field extensions of k_0, and the arrows are morphisms of k_0-algebras. We think of $\mathrm{H}^\bullet$ as a functor from (Field/k_0) to the category of graded-commutative $\mathbb{Z}/p$-algebras.

Assume that we are given a functor $F : (\mathrm{Field}/k_0) \to (\mathrm{Set})$. A cohomological invariant of F is a natural transformation $F \to \mathrm{H}^\bullet$. The cohomological invariants of F form a graded-commutative ring $\mathrm{Inv}^\bullet(F)$.

Given an algebraic group G, one can define the cohomological invariants of G as $\mathrm{Inv}^\bullet(\mathit{Tors}_G)$, where Tors_G is the functor sending each extension K of k_0 to the set of isomorphism classes of G-torsors over $\mathrm{Spec}(K)$. The book [GMS03] is dedicated to the study of cohomological invariants of algebraic groups; since many algebraic structures correspond to G-torsors for various groups G (some of the best known examples are étale algebras of degree n corresponding to S_n-torsors, nondegenerate quadratic forms of rank n corresponding to O_n-torsors, and central simple algebras of degree n corresponding to PGL_n-torsors) this gives a unified approach to the cohomological invariants for various types of structures.

II Cohomological invariants of smooth algebraic stacks

Suppose that $\mathscr{M}$ is an algebraic stack smooth over k_0, for example the stack $\mathscr{M}_g$ of smooth curves of genus g for $g \geq 2$. We can define a functor $F_{\mathscr{M}} : (\mathrm{Field}/k_0) \to (\mathrm{Set})$ by sending a field K to the set of classes of isomorphism in $\mathscr{M}(K)$, and thus a ring $\mathrm{GInv}^\bullet(\mathscr{M})$ of *general* cohomological invariants of $\mathscr{M}$ defined as natural transformations between $F_{\mathscr{M}}$ and $\mathrm{H}^\bullet$.

The definition above recovers the definition of cohomological invariants of an algebraic group G when $\mathscr{M} = BG$, the stack of G-torsors. However, when the objects of $\mathscr{M}$ are not étale locally isomorphic, as in the case of BG, this is not the right notion. For example, if $\mathscr{M}$ has a moduli space M every object of $\mathscr{M}(K)$ determines a point $p \in M$, corresponding to the composite $\mathrm{Spec}(K) \to \mathscr{M} \to M$. If we denote $F^p_{\mathscr{M}}$ the subfunctor of $F_{\mathscr{M}}$ corresponding to isomorphism classes of objects in $\mathscr{M}$ with image p it is easy to see that

$$\mathrm{GInv}^{\bullet}(F_{\mathscr{M}}) = \prod_{p \in M} \mathrm{GInv}^{\bullet}(F^p_{\mathscr{M}})$$

This is clearly too large to be interesting. We need to impose a continuity condition to be able to compare cohomological invariants at points of $\mathscr{M}$ with different images in M.

The following condition turns out to be the correct one when $\mathscr{M}$ is smooth. Let R be an Henselian k_0-algebra that is a discrete valuation ring, with fraction field K and residue field k. We have induced cohomology maps $\mathrm{H}^{\bullet}(R) \to \mathrm{H}^{\bullet}(k)$ and $\mathrm{H}^{\bullet}(R) \to \mathrm{H}^{\bullet}(K)$; the first is well-known to be an isomorphism [Sta15, 04GE]. By composing the second map with the inverse of the first we obtain a ring homomorphism $j_r : \mathrm{H}^{\bullet}(k) \to \mathrm{H}^{\bullet}(K)$. Furthermore, from an object $\xi \in \mathscr{M}(R)$ we obtain objects $\xi_k \in \mathscr{M}(k)$ and $\xi_K \in \mathscr{M}(K)$.

Definition 1. A general cohomological invariant $\alpha \in \mathrm{GInv}(\mathscr{M})$ is *continuous* if for every Henselian DVR as above, and every $\xi \in \mathscr{M}(R)$ we have

$$j(\alpha(\xi_K)) = \alpha(k).$$

Continuous cohomological invariants form a graded subring $\mathrm{Inv}^{\bullet}(\mathscr{M})$ of $\mathrm{GInv}(\mathscr{M})$.

When $\mathscr{M}$ is equal to BG all (general) cohomological invariants are continuous (this is a result by Rost [GMS03, 11.1]). Thanks to this from now on we will be able to abuse notation and forget about the word continuous without being cause for confusion.

If $\mathscr{M} = X$, where X is a scheme, the cohomological invariants of X can be described as the Zariski sheafification of the presheaf $U \to \mathrm{H}^{\bullet}(U)$, which is in turn equal to the unramified cohomology of X (see [BO74, 4.2.2]).

The Zariski topology is clearly too coarse for a similar result to hold for algebraic stacks. The appropriate class of morphisms to study turns out to be the following.

Definition 2. Let $\mathscr{M}, \mathscr{N}$ be algebraic stacks. A *smooth-Nisnevich* covering $f : \mathscr{M} \to \mathscr{N}$ is a smooth representable morphism such that for ever field K and every object $\xi \in \mathscr{N}(K)$ we have a lifting

$$\mathrm{Spec}(K) \xrightarrow{\xi'} \mathscr{M} \xrightarrow{f} \mathscr{N}, \quad f \circ \xi' \simeq \xi.$$

Cohomological invariants satisfy the sheaf conditions with respect to smooth-

Nisnevich morphisms. Namely, given a smooth-Nisnevich morphism $f : \mathscr{M} \to \mathscr{N}$ the cohomological invariants $\mathrm{Inv}^\bullet(\mathscr{N})$ are the equalizer of the following diagram

$$\mathrm{Inv}^\bullet(\mathscr{N}) \xrightarrow{f^*} \mathrm{Inv}^\bullet(\mathscr{M}) \underset{\mathrm{Pr}_2{}^*}{\overset{\mathrm{Pr}_1{}^*}{\rightrightarrows}} \mathrm{Inv}^\bullet(\mathscr{M} \times_{\mathscr{N}} \mathscr{M})$$

Using this and the result for schemes we obtain the following:

Theorem 1. *Let $\mathscr{M}$ be an algebraic stack smooth over k_0. Denote $\mathscr{C}$ to be the site of $\mathscr{M}$-schemes with covers given by Nisnevich morphisms. Consider the sheafification $\mathscr{H}^\bullet$ of the presheaf $U \to \mathrm{H}^\bullet(U)$ in $\mathscr{C}$. Then*

$$\mathrm{Inv}^\bullet(\mathscr{M}) = H^0(\mathscr{M}, \mathscr{H}^\bullet).$$

From this we obtain the fundamental corollary

Corollary 1.

1. *Let $\mathscr{E} \to \mathscr{M}$ be a vector bundle. Then the pullback $\mathrm{Inv}^\bullet(\mathscr{M}) \to \mathrm{Inv}^\bullet(\mathscr{E})$ is an isomorphism.*

2. *Let $\mathscr{N}$ be a closed substack of codimension 2 or more. Then the pullback $\mathrm{Inv}^\bullet(\mathscr{M}) \to \mathrm{Inv}^\bullet(\mathscr{M} \smallsetminus \mathscr{N})$ is an isomorphism.*

If $\mathscr{M}$ is a smooth quotient stack, that is, a quotient $[X/G]$, where G is an affine algebraic group over k_0 acting over a smooth algebraic space X, then there exists a vector bundle $\mathscr{E} \to [X/G]$ with an open substack $V \to \mathscr{E}$, where V is an algebraic space, such that the complement of V in $\mathscr{E}$ has codimension at least 2 [EG96, 7,Appendix]. Then $\mathrm{Inv}^\bullet([X/G]) = \mathrm{Inv}^\bullet(V)$ (in the case of $\mathscr{M} = BG$ this result was obtained by Totaro in [GMS03, Appendix C]). For many purposes, this allows to reduce the study of cohomological invariants of general smooth algebraic stacks to the case of smooth algebraic spaces.

The main theorem fails as soon as $\mathscr{M}$ is no longer normal as the continuity condition fails to provide sufficient constraints on the invariants. It is possible that some of the theory may works for some intermediate condition between the two.

The last part of the first chapter is dedicated to extending the notion of cohomological invariants to allow for coefficients in any functor that satisfies the property of Rost's cycle modules [Ros96]. The main example for these, besides étale cohomology, is Milnor's K-theory.

The price to pay for the generalization is a slightly less clear characterization of cohomological invariants. As the isomorphism between the étale cohomology of a Henselian DVR (R, v) and that of its residue field is no longer true, we do not have any natural map $M(k(v)) \to M(k(R))$, and instead we use a quotient of $M(k(R))$ which Rost denotes $M(v)$. There is a canonical projection $p : M(k(R)) \to M(v)$, and a natural inclusion $i : M(k(v)) \to M(v)$, so we can reformulate the continuity condition in the following way

Definition 3. Let $\mathscr{X}$ be an algebraic stack, and M a cycle module. A cohomological invariant for $\mathscr{X}$ with coefficients in M is a natural transformation from the functor of points

$$F_{\mathscr{X}} : (\text{field}/k_0) \to (\text{set})$$

to M, satisfying the following property: given a Henselian DVR R and a map $\operatorname{Spec}(R) \to \mathscr{X}$ we have $p(\alpha(k(R)) = i(\alpha(k(v))$ in $M(v)$.

When $M = \mathrm{H}^\bullet$, the definition is equivalent to the original one. The main theorem still holds, using Rost's 0-codimensional Chow group with coefficients in M in place of the sheaf $\mathscr{H}$.

Theorem 2. *Let $\mathscr{X}$ be an algebraic stack smooth over k_0, and let M be a cycle module. Denote $\mathscr{C}$ to be the site of $\mathscr{M}$-schemes with covers given by Nisnevich morphisms. Consider the sheafification $\mathscr{H}^\bullet$ of the presheaf $U \to \mathrm{H}^\bullet(U)$ in $\mathscr{C}$. Then*

$$\operatorname{Inv}^\bullet(\mathscr{X}, M) = H^0(\mathscr{X}, A^0(-, M)).$$

III Chow rings with coefficients

In his paper [Ros96], Markus Rost defined a generalization of ordinary Chow groups, whose coefficients, rather than being integers, depend on the choice of a functor from fields to abelian groups satisfying a certain set of conditions. The two main examples of such functors, which he calls "cycle modules", are Milnor's K-theory and Galois cohomology.

Remarkably, Rost's Chow groups with coefficients not only retain all of the powerful functorial properties of ordinary Chow groups, they also convey both informations on the ordinary Chow groups (in the case of Milnor's K-theory, complete information) and more cohomological information such as, in the case of Galois cohomology, the full unramified cohomology, as the 0-codimensional group $A^0(X; \mathrm{H}^\bullet)$ is by definition equal to $\mathrm{H}_{nr}(X)$. Moreover, if X is smooth Rost's Chow groups with coefficients form a ring $A^\bullet(X; \mathrm{H}^\bullet)$.

The idea of using Chow groups with coefficients to compute cohomological invariants is developed in [Gui08], where they are used in conjunction with the stratification method first developed in [Vez00] to give an alternative way of computing the cohomological invariants of various classical groups and to obtain new information on the cohomological invariants of the spin groups.

Rost's theory creates a perfect framework for the computations of cohomological invariants, but there are two slight extensions needed. First, we want the theory to work for algebraic spaces. As our main aim is to compute the cohomological invariants of quotient stacks, we want to avoid altogether the technicalities needed to ensure that the approximations we use are schemes rather than just algebraic spaces, and in general we want to avoid having to consider gluing conditions as much as possible, as they can be very difficult to read, so having a functional theory for a larger category of objects is needed. Last but not least, we believe that an extension of the theory to algebraic spaces is in general desirable to match the generality available for ordinary Chow groups.

The extension to algebraic spaces is obtained by passing from the Zariski topology to the Nisnevich topology, and consequently from local rings to local Henselian rings. The verifications to be made are generally simple, although many and sometimes lengthy.

We have also tried to clarify and make more explicit parts of the original paper, especially the last chapters, where many important properties of the intersection pairing are only hinted.

The second extension we need is an operational theory of Chern classes. This has been done when the coefficients are taken in Milnor's K-theory in [EKM08, Chap 9]. We will extend the theory to all cycle modules and add a few results that are needed for our computations in part 2.

We conclude with some remarks on the equivariant version of Chow groups with coefficients.

IV Cohomological invariants of hyperelliptic curves

The theory we set up in Chapters 1 and 2 can be used to compute the cohomological invariants of the stacks of Hyperelliptic curves of all even genera, and of genus three. The main result is the following:

Theorem 3. *Suppose our base field k_0 is algebraically closed, of characteristic different from $2, 3$.*

- *Suppose g is even. For $p = 2$, the cohomological invariants of $\mathscr{H}_g$ are freely generated as a graded $\mathbb{F}_2$-module by 1 and invariants $x_1, \ldots, x_{g+2}$, where the degree of x_i is i.*

  If $p \neq 2$, then the cohomological invariants of $\mathscr{H}_g$ are nontrivial if and only if $2g + 1$ is divisible by p. In this case they are freely generated by 1 and a single invariant of degree one.

- *Suppose $g = 3$. For $p = 2$ the cohomological invariants of $\mathscr{H}_3$ are freely generated as a $\mathbb{F}_2$-module by 1 and elements $x_1, x_2, w_2, x_3, x_4, x_5$, where the degree of x_i is i and w_2 is the second Stiefel-Whitney class coming from the cohomological invariants of PGL_2.*

  If $p \neq 2$, then the cohomological invariants of $\mathscr{H}_3$ are trivial for $p \neq 7$ and freely generated by 1 and a single invariant of degree one for $p = 7$.

Moreover, we obtain partial results for general fields, proving in general that the cohomological invariants above are freely generated as a $\mathrm{H}^\bullet(k_0)$-module by the same elements if p differs from 2. If p is equal to 2, we show that the cohomological invariants of $\mathscr{M}_2$ and $\mathscr{H}_3$ are the direct sum of a freely generated $\mathrm{H}^\bullet(k_0)$-module, whose generators are the same as in the algebraically closed case except for the one of highest degree, and a module K, which is a submodule of $H^\bullet(k_0)[4]$ or $H^\bullet(k_0)[5]$ respectively.

Our method is based on the presentation by Vistoli and Arsie [AV04, 4.7] of the stacks of hyperelliptic curves as the quotient of an open subset of an affine space by a group G, which is equal to GL_2 for even genera and $PGL_2 \times G_m$ for odd genera. We use a technique similar to the stratification method introduced by Vezzosi in [Vez00] and used by various authors afterwards ([Gui08],[VM06]). The idea is, given a quotient stack $[X/G]$, to construct a stratification $X = X_0 \supset X_1 \ldots \supset X_n = \emptyset$ of X such that the geometry of $X_i \smallsetminus X_{i+1}$ is simple enough that we can compute inductively the invariants for X_i using the result for X_{i+1} and the localization exact sequence [Ros96, p. 356].

One flaw of our method of computation is that it does not provide any real insight on the product in the ring of cohomological invariants of $\mathscr{H}_g$. The reason is that repeatedly applying the localization exact sequence causes loss of information about our elements, making it very difficult to understand what their products should be.

V Description of content

V.1 Chapter 1

The first chapter is dedicated to giving an explicit description of the ring of cohomological invariants. After we give the definition, the smooth-Nisnevich topology is defined on various sites related to an algebraic stack $\mathscr{M}$, and we prove that the topoi arising from these sites are all equivalent. In particular, on schemes the smooth-Nisnevich topology is equivalent to the usual Nisnevich topology. We prove that the ring of cohomological invariants is a sheaf on all of these sites.

The rest of the chapter is dedicated to proving Theorem 1, which is equivalent to saying that the cohomological invariants are the sheafification of étale cohomology in the smooth-Nisnevich topology for smooth stacks. To do this we first prove that for a smooth irreducible scheme a cohomological invariant is determined by its value at the generic point, and then apply the well-know description of unramified cohomology given by the Gersten resolution [BO74, 4.2.2]. As a corollary of our description we obtain that cohomological invariants are conserved by affine bundles and by removing a closed substack of codimension two or more.

To provide a first application our techniques, we compute the cohomological invariants of the stack of elliptic curves $\mathscr{M}_{1,1}$. It turns out that the only nontrivial invariant is the one sending an elliptic curve with Weierstrass form $y^2 = x^3 + ax + b$ to the element $[4a^3 + 27b^2] \in k^*/k^{*p} \simeq \mathrm{H}^1(\mathrm{Spec}(k))$, when $p = 2$ or $p = 3$.

Lastly we extend our definition to allow for invariants taking values in arbitrary cycle modules. After making a few technical points to give a new continuity condition and show that it is equivalent to the former for étale cohomology, we prove that that the results in the chapter hold almost verbatim in this extended setting.

V.2 Chapter 2

The first section of the chapter is dedicated to reworking the theory of Chow groups with coefficients for algebraic spaces. The main step here is moving from the Zariski topology to the Nisnevich topology, which is possible thanks to [Knu71, 6.3], which assures us that a quasi-separated algebraic space has a Nisnevich cover by a scheme. The rest of the work consists in carefully reducing our proofs to the case of schemes. We tried to give a systematic treatment of the subject, which could provide an introduction to the subject for those readers that are not familiar with the original paper [Ros96].

The second section is dedicated to constructing a theory of Chern classes and giving a quick overview of the equivariant theory, which was first introduced in [Gui08]. The first Chern class of a line bundle $E \to X$ is constructed explicitly as the composition of the zero section and the retraction $r : C_\bullet(E) \to C_\bullet(X)$ constructed in [Ros96, sec.9]. General Chern classes are constructed in the same way as in [Ful84, Ch.3], using Segre classes. All the basic formulas for Chern classes hold with coefficients, and we use them to compute the Chow rings with coefficients of Grassmann bundles.

The last part of the section is dedicated to the equivariant version of the theory. To see that the whole theory translates to the equivariant setting we can just repeat the proofs used for the ordinary Chow groups in [EG96]. We use the results on Grassmanian bundles to compute the Chow rings with coefficients of GL_n and SL_n, and we extend the result in [Ros96, 6.5] stating that Chow groups with coefficients are equal to the Zariski cohomology of the sheafification of $(U \to A^0(U))$ by proving that for a smooth algebraic space X with an action of a smooth algebraic group G we have

$$A^\bullet_G(X) \simeq H^\bullet_{sm-Nis}([X/G], \mathscr{A})$$

where $\mathscr{A}$ is the smooth-Nisnevich sheafification of $(U \to A^0(U))$.

V.3 Chapter 3

The third chapter is dedicated to computing the cohomological invariants with mod p coefficients for the stacks of hyperellptic curves of various genera.

In the first section we describe the presentation of the stacks $\mathscr{H}_g$ as quotients $[U_g/G]$. If we see $\mathbb{A}^{2g+3}$ as the space of binary forms of degree $2g+2$, the scheme $U_g \subset \mathbb{A}^{2g+3}$ is the open subscheme of nonzero forms with distinct roots. We show that cohomological invariant can be computed on the projectivized space $Z_g = X_g/G_m$, where G_m acts by multiplication, and we introduce a stratification $P^{2g+2} \supset \Delta_{1,2g+2} \supset \ldots \supset \Delta_{g+1,2g+2}$ which will be the base of our computation. We can see $\Delta_{i,2g+2}$ as the closed subscheme of binary forms divisible by the square of a form of degree i, and we have $Z_g = P^{2g+2} \smallsetminus \Delta_{1,2g+2}$.

The second section is dedicated to computing the cohomological invariants of $\mathscr{M}_2$ over an algebraically closed field. The proof is kept as elementary as possible to give the reader a gradual introduction to the techniques used, and using the fact that we are working over an algebraically closed field everything is worked out with considerations on the equivariant Chow groups mod p. The argument is based on the fact that the equivariant Chow groups with coefficients of $\Delta_{i,n} \smallsetminus \Delta_{i+1,n}$ are isomorphic to those of $P^{n-2i} \smallsetminus \Delta_{1,n-2i} \times P^i$, giving rise to an inductive reasoning.

In the third section we compute the invariants for $\mathscr{H}_g$ for all even g. The techniques used for $\mathscr{M}_2$ are improved and generalized as more specific arguments involving the functoriality of the localization exact sequence and the projection formula are introduced.

The fourth section is dedicated to extending the previous results to fields that are not algebraically closed. The extension turns out to be immediate when the prime p is different from 2, and rather troublesome for $p = 2$. The main difficulty lies in understanding if the pushforward through the closed immersion $\Delta_{1,6} \to P^6$ induces the zero map on $A^0_{GL_2}$. To do so, we construct an element in degree 0 which belongs to the annihilator of the image of $A^0_{GL_2}(\Delta_{1,6})$ but does not belong to the annihilator of any nonzero element of $A^1_{GL_2}(P^6)$. This has to be done at cycle level, and the construction relies heavily on the explicit description of the first Chern class of a line bundle given in chapter 2.

In the fifth section we compute the equivariant Chow groups with coefficients of the classifying spaces of μ_p, O_2, O_3 and SO_3. This is needed as $\mathscr{H}_3$ is described as a quotient by an action of $PGL_2 \times G_m$, and the equivariant Chow ring $A^\bullet_{SO_3}(\mathrm{Spec}(k_0))$ is isomorphic to $A^\bullet_{PGL_2}(\mathrm{Spec}(k_0))$. We follow step by step the stratification method used in [VM06], with some minor changes. The computations are done both with coefficients in étale cohomology and in Milnor's K-theory, as the proofs can be adapted easily.

In the sixth section we compute the cohomological invariants of $\mathscr{H}_3$. The fact that PGL_2 is not special creates some additional complications, as the equivariant ring $A^\bullet_{PGL_2}(\mathrm{Spec}(k_0))$ has several nonzero elements in positive degree even for an algebraically closed k_0. The main difficulty is again proving that the map $A^0_{PGL_2}(\Delta_{1,n}) \to$

$A^i_{PGL_2}(P^n)$ is zero. Luckily the richer structure of $A^\bullet_{PGL_2}(\mathrm{Spec}(k_0))$ comes to our aid, and we are able to inductively construct an element f_n in the annihilator of the image of $A^0_{PGL_2}(\Delta_{1,n})$ which for $n \leq 8$ does not belong to the annihilator of any nonzero element in $A^1_{PGL_2}(P^n)$, allowing us to conclude.

1. COHOMOLOGICAL INVARIANTS

In this chapter we give the definition of cohomological invariants and prove their main properties.

The first section is where the actual definition is given, along with some initial remarks. In the second section we explore the smooth-Nisnevich site of an algebraic stack, which is the natural context where cohomological invariants are studied.

The third section is dedicated to showing that cohomological invariants are the sheafification of unramified cohomology on the smooth-Nisnevich site, which has the immediate corollaries of them being invariant by passing to affine bundles and by modifications in codimension greater than one.

Finally in the fourth section we extend the notion of cohomological invariants by allowing their values to take place in more general functors.

1 Preliminaries

In this chapter we fix a base field k_0 *and a prime number* p*. We will always assume that the characteristic of* k_0 *is different from* p*. All schemes and algebraic stacks will be assumed to be of finite type over* k_0*. If* X *is a* k_0*-scheme we will denote by* $\mathrm{H}^i(X)$ *the étale cohomology ring of* X *with coefficients in* $\mu_p^{\otimes i}$ *(here* $\mu_p^{\otimes 0} := \mathbb{Z}/p\mathbb{Z}$*), and by* $\mathrm{H}^\bullet(X)$ *the direct sum* $\oplus_i \mathrm{H}^i(X)$*. If* R *is a* k_0*-algebra, we set* $\mathrm{H}^\bullet(R) = \mathrm{H}^\bullet(\mathrm{Spec}(R))$*.*

We begin by defining the notion of a cohomological invariant and exploring some of its consequences.

Lemma 1.1. *Let* R *be an Henselian ring,* k *its residue field. The closed immersion* $\mathrm{Spec}(k) \to \mathrm{Spec}(R)$ *induces an isomorphism of graded rings* $\mathrm{H}^\bullet(R) \to \mathrm{H}^\bullet(k)$*.*

Proof. This is Gabber's theorem, [Sta15, 09ZI]. □

Given a Henselian ring R, with residue field k and field of fractions K, we can construct a map

$$j : \mathrm{H}^\bullet(\mathrm{Spec}(k)) \to \mathrm{H}^\bullet(\mathrm{Spec}(K))$$

by composing the inverse for the isomorphism $\mathrm{H}^\bullet(\mathrm{Spec}(R)) \to \mathrm{H}^\bullet(\mathrm{Spec}(k))$ and the pullback $\mathrm{H}^\bullet(\mathrm{Spec}(R)) \to \mathrm{H}^\bullet(\mathrm{Spec}(K))$. The same map j is obtained by different methods in [GMS03, 7.6,7.7].

Definition 1.2. Let $\mathscr{M}$ be an algebraic stack. A *cohomological invariant* of $\mathscr{M}$ is a natural transformation

$$\alpha : {}^{\mathrm{Hom}(-,\mathscr{M})}\!/_{\cong} \to \mathrm{H}^\bullet(-)$$

seen as functors from $\left({}^{\mathit{field}}\!/_{k_0}\right)$ to (set), satisfying the following property:
Let X be the spectrum of a Henselian DVR, p, P its closed and generic points. Then given a map $f : X \to \mathscr{M}$, we have

$$\alpha(f \circ P) = j(\alpha(f \circ p)). \tag{1.1}$$

The grading and operations on cohomology endow the set of cohomological invariants of $\mathscr{M}$ with the structure of a graded ring, which we will denote $\mathrm{Inv}^\bullet(\mathscr{M})$.

As stated in the introduction, if $\mathscr{M}$ is the stack of G torsors for an algebraic group G condition 1.1 is automatic. This is proven in [GMS03, 11.1]. The continuity condition has the very important consequence of tying cohomological invariants to another well-known invariant, unramified cohomology.

Definition 1.3. Let K be a field. Given a discrete valuation v on K there is a standard residue map $\partial_v : \mathrm{H}^\bullet(K) \to \mathrm{H}^\bullet(k(v))$ of degree -1 (for example it is constructed in [GMS03, sec.6-7]).

Consider a field extension K/k. The *unramified cohomology* $\mathrm{H}^\bullet_{nr}(K/k)$ is the kernel of the map

$$\partial : \mathrm{H}^\bullet(K) \to \bigoplus_v \mathrm{H}^\bullet(k(v))$$

defined as the direct sum of δ_v over all discrete valuations on K that are trivial on k. Given a normal scheme X over k we define the unramified cohomology $\mathrm{H}^\bullet_{nr}(X)$ to be the kernel of the map

$$\partial_X : \mathrm{H}^\bullet(K) \to \bigoplus_{p \in X^{(1)}} \mathrm{H}^\bullet(k(v_p))$$

here the sum is over all valuations induced by a point of codimension one. For an irreducible scheme X we define $\mathrm{H}^\bullet_{nr}(X)$ to be the unramified cohomology of its normalization. If R is a k_0-algebra we denote $\mathrm{H}^\bullet_{nr}(R) = \mathrm{H}^\bullet_{nr}(\mathrm{Spec}(R))$.

When X is an irreducible and normal scheme, proper over $\mathrm{Spec}(k)$, we have the equality $\mathrm{H}^\bullet_{nr}(X) = \mathrm{H}^\bullet_{nr}(k(X)/k)$, making unramified cohomology a birational invariant. It was first introduced to study rationality problems in the guise of the unramified Brauer group [Sal84], and then generalized to all degrees [CTO89].

Remark 1.4. Let X be the spectrum of a DVR R, with residue field k and fraction field K. The residue map $\partial_v : \mathrm{H}^\bullet(K) \to \mathrm{H}^\bullet(k)$ has the property that if we consider the map $X^h \xrightarrow{h} X$ induced by the Henselization of R, and the residue map $\partial_{v'}$ on X^h we have (see [Ros96, sec 1, R3a])

$$\partial_{v'} \circ h* = \partial_v : \mathrm{H}^\bullet(P) \to \mathrm{H}^\bullet(p).$$

As we have $\partial_{v'} \circ j = 0$ [GMS03, 7.7], we conclude that given a normal scheme X and a cohomological invariant $\alpha \in \mathrm{Inv}^\bullet(X)$ the value of α at the generic point ξ_X belongs to $\mathrm{H}^\bullet_{nr}(X)$.

In fact we can say much more. There is an obvious pullback on cohomological invariants:

Definition 1.5. Given a morphism $f : \mathscr{M} \to \mathscr{N}$, we define the pullback morphism $f^* : \mathrm{Inv}^\bullet(\mathscr{N}) \to \mathrm{Inv}^\bullet(\mathscr{M})$ by setting $f(\alpha)(p) = \alpha(p \circ f)$.

Given any map $f : X \to \mathscr{M}$ from an irreducible scheme X, with generic point ξ_X, consider the pullback of a cohomological invariant $\alpha \in \mathrm{Inv}(\mathscr{M})$. We can apply the remark above to $f^*(\alpha)$, obtaining $\alpha(\xi_X) = f^*(\alpha)(\xi_X) \in \mathrm{H}^\bullet_{nr}(X)$. In the case of a closed immersion $V \to X$, with V irreducible, this says that the value $\alpha(\xi_V)$ belongs to the unramified cohomology $\mathrm{H}^\bullet_{nr}(V)$.

It is immediate to check that the cohomological invariants of the spectrum of a field are canonically isomorphic to its étale cohomology. Moreover, there is a natural map from étale cohomology to cohomological invariants sending an element $x \in \mathrm{H}^\bullet(\mathscr{M})$ to the invariant $\tilde{x}$ defined by $\tilde{x}(p) = p^*(x)$. The étale cohomology of an algebraic stack is defined as the sheaf cohomology in its Lisse-étale site [Sta15, 01FQ, 0786].

If R, k, K are as in the definition the elements $\tilde{x}(p), \tilde{x}(P)$ are both pullbacks of $f^*(x) \in \mathrm{H}^\bullet(R)$, and the functoriality of pullback allows us to conclude that the continuity condition (1.1) is fulfilled. This map is clearly not injective, as the next example shows.

Example 1.6. An easy example of the map $\mathrm{H}^\bullet(\mathscr{M}) \to \mathrm{Inv}^\bullet(\mathscr{M})$ not being injective comes from computing the cohomology ring of $\mathrm{B}(\mathbb{Z}/2)$ with coefficients in $\mathbb{F}_2$ over an algebraically closed field of characteristic different from 2. By the Hochschild-Serre spectral sequence and group cohomology we obtain $\mathrm{H}^\bullet(\mathrm{B}(\mathbb{Z}/2)) = \mathbb{F}_2[t]$, where $\deg(t) = 1$, while $\mathrm{Inv}(\mathrm{B}(\mathbb{Z}/2), p) = \mathbb{F}_2[t]/t^2$ when $p = 2$ [GMS03, 16.2].

2 The smooth-Nisnevich sites

We want to make $\mathrm{Inv}^\bullet$ into a sheaf for an appropriate Grothendieck topology. It cannot be a sheaf in the étale topology as the pullback through a finite separable extension is not in general injective for étale cohomology. The Zariski topology is not satisfactory as algebraic stacks do not have Zariski covers by schemes. The Nisnevich topology, consisting of étale morphisms $X \to Y$ having the property that any map from the spectrum of a field to Y lifts to X looks like a promising way between, at least for Deligne-Mumford stacks. Unfortunately it still does not fit our needs, as the following example shows:

Example 2.1. There are Deligne-Mumford stacks that do not admit a Nisnevich covering by a scheme, and in this is a very common occurrence.

Consider $\mathscr{M} = B(\mu_2)$. The μ_2-torsor $P = \mathrm{Spec}(k(t)) \to \mathrm{Spec}(k(t^2))$ is not obtainable as the pullback of any torsor $T \to \mathrm{Spec}(k)$ with k finite over k_0. This shows that given an étale map $X \to \mathscr{M}$ there cannot be a lifting of P to X, as any point of X will map to a torsor T as above.

This is a general problem tied to the essential dimension ([BR97],[BF03]): whenever we have a strict inequality $\dim(\mathscr{M}) < \mathrm{ed}(\mathscr{M})$ there can be no Nisnevich cover of $\mathscr{M}$ by a scheme. For an irreducible Deligne-Mumford stack this happens whenever the generic stabiliser is not trivial.

To solve this problem, we admit all smooth maps satisfying the lifting property:

Definition 2.2. Let $f : \mathscr{M} \to \mathscr{N}$ be a representable morphism of algebraic stacks, $p \in \mathscr{N}(K)$. Then f is a *Smooth-Nisnevich* neighbourhood of p if it is smooth and there is a representative $\mathrm{Spec}(K) \to \mathscr{N}$ of the isomorphism class of p such that we have a lifting

$$\begin{array}{ccc} & & \mathscr{M} \\ & \overset{p'}{\nearrow} & \downarrow f \\ \mathrm{Spec}(k) & \xrightarrow{p} & \mathscr{N} \end{array}$$

f is a *Smooth-Nisnevich* cover if for every field K and every $p \in \mathscr{N}(K)$ it is a *Smooth-Nisnevich* neighbourhood of p.

Note that if f is a *Smooth-Nisnevich neighbourhood* of p, then given any representative of p such lifting exists.

This topology looks awfully large. Luckily, as we will prove shortly, when restricted to schemes it coincides with the usual Nisnevich topology. Recall that given a quasi-separated algebraic space we always have an étale Nisnevich cover by a scheme [Knu71, 6.3], so we can trivially extend the Nisnevich topology to the category of quasi-separated algebraic spaces.

Proposition 2.3. *Name* $\mathrm{Alg}/_{k_0}$ *the category of quasi-separated algebraic spaces over the spectrum of* k_0. *Let* F *be a presheaf on* $\mathrm{Alg}/_{k_0}$. *The sheafification of* F *with respect to the Ninsnevich topology is the same as its sheafification with respect to the Smooth-Nisnevich topology.*

Proof. We need to show that any smooth-Nisnevich cover has a section Nisnevich locally. As we can take a Nisnevich cover of an algebraic space that is a scheme, we can restrict to schemes. Recall (see [Liu02, ch.6,2.13-2.14]) that if $f : X \to \mathrm{Spec}(R)$ is a smooth morphism from a scheme to the spectrum of an Henselian ring with residue field k, given a k-rational point p of X there is always a section of f sending the closed point of $\mathrm{Spec}(R)$ to p. Let now $X \to Y$ be a smooth-Nisnevich cover. Consider the diagram:

$$\begin{array}{ccc} X \times_Y \mathrm{Spec}(\mathscr{O}^h_{Y,q}) & \xrightarrow{\mathrm{pr}_1} & X \\ \downarrow \mathrm{pr}_2 & & \downarrow f \\ \mathrm{Spec}(\mathscr{O}^h_{Y,q}) & \xrightarrow{j} & Y \end{array}$$

As q lifts to a point of X, the left arrow has a section. The scheme $\mathrm{Spec}(\mathscr{O}^h_{Y,q})$ is the direct limit of all the Nisnevich neighbourhoods of q, so there is a Nisnevich neighbourhood U_q of q with a lifting to X. By taking the disjoint union over the points of X we obtain the desired Nisnevich local section. □

In particular, the local ring at a point of a scheme in the smooth-Nisnevich topology is still the Henselization of the local ring in the Zariski topology, and in general if we consider a category of algebraic spaces containing all étale maps the topoi induced by the Nisnevich and smooth-Nisnevich topology will be equivalent.

The smooth-Nisnevich topology has the annoying problem that an open subset of a vector bundle may not be a covering when working over finite fields. To solve this, we introduce the following larger topologies.

Definition 2.4. Let $f : \mathscr{M} \to \mathscr{N}$ be a representable morphism of algebraic stacks, $p \in \mathscr{N}(K)$. Let m be a non negative integer. Then f is a *m-Nisnevich* (resp. *Smooth m-Nisnevich*) neighbourhood of p if it is étale (resp. smooth) and there are finite separable extensions $K_1, \ldots, K_r$ of K with liftings

$$\begin{array}{ccc} \mathrm{Spec}(K_i) & \xrightarrow{\;p'\;} & \mathscr{M} \\ \big\downarrow{\scriptstyle \phi_i} & & \big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(k) & \xrightarrow{\;p\;} & \mathscr{N} \end{array}$$

And such that $([K_1 : K], \ldots, [K_r : K], m) = 1$.

f is an *m-Nisnevich* (resp. *Smooth m-Nisnevich*) cover if for every field K and every $p \in \mathscr{N}(K)$ it is an *m-Nisnevich* (resp. *Smooth m-Nisnevich*) neighbourhood of p.

The m-Nisnevich topology strictly contains the Nisnevich topology for all m. If $m = 1$, we get the étale topology, and the m-Nisnevich topology contains the n-Nisnevich topology if and only if the prime factors of n divide m.

Proposition (2.3) holds *verbatim* for the m-Nisnevich and smooth m-Nisnevich topologies, as we can just repeat the argument adding some base changes.

Lemma 2.5. *Let $V \to \mathrm{Spec}(k)$ be a vector space, and $U \to V$ a non-empty open subset. Then U is an arithmetic smooth-Nisnevich cover of* $\mathrm{Spec}(k)$*, and if k is infinite it is a smooth-Nisnevich cover.*

Proof. It suffices to prove this this for a $V = \mathbb{A}^1_k$. The statement for k infinite is obvious. Suppose now that k is finite. A closed subset $Z \subsetneq V$ only contains a finite number of closed point, so for any prime q we can always find points p_q with $[k(p_q) : k] = q^n$ for n large enough, implying the result. □

Proposition 2.6. *Let $\mathscr{M}$ be a quasi-separated algebraic stack. There exists a countable family of algebraic spaces X_n with maps $p_n : X_n \to \mathscr{M}$ of finite type such that the union of these maps is a smooth-Nisnevich cover.*

If $\mathscr{M}$ has affine stabilizers group at all of its geometric points, and the base field is infinite, we only need a finite number of maps $p_n : X_n \to \mathscr{M}$.

In general if $\mathscr{M}$ has affine stabilizers group at all of its geometric points, we only need a finite number of maps to obtain a smooth m-Nisnevich cover.

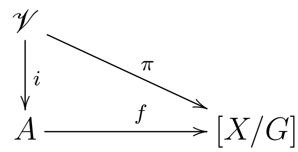
Proof. The first statement is proven in [LMB99, 6.5]. Note that by dropping the geometrically connectedness request on the fibres we can extend the covering family to the whole stack.

Let now $\mathscr{M}$ be an algebraic stack with affine stabilizers groups. We want to prove that there is an object $P : \mathrm{Spec}(K) \to \mathscr{M}$ of $\mathscr{M}$ that is *versal*, meaning that for any smooth morphism $f : X \to \mathscr{M}$ from an algebraic space to $\mathscr{M}$ with a lifting of P to f there is an open substack $\mathscr{U}$ of $\mathscr{M}$ with the property that every point of $\mathscr{U}$ lifts to f.

In [Kre99, 3.5.9] Kresch proves that under the hypothesis of affine stabilizer groups an algebraic stack admits a stratification by quotient stacks $[X/G]$, where X is an algebraic space and G a linear algebraic group. We may thus suppose that we are working in these hypotheses. Moreover we may suppose our stack is irreducible.

For a quotient stack we have a standard approximation by an algebraic space. Let V be a representation of G such that there is an open subset U of V on which G acts freely, and $V \setminus U$ has codimension two or more in V. Then $[(X \times V)/G] \xrightarrow{\pi} [X/G]$ is a vector bundle, and $[(X \times U)/G]$ is an algebraic space. Note that given any open subset $\mathscr{V}$ of $[(X \times V)/G]$, the restriction of π is a smooth-Nisnevich cover of some open substack $\mathscr{U}$ of $[X/G]$. First we reduce to an open subset $\mathscr{U}$ of $[X/G]$ such that the fiber of all points of $\mathscr{U}$ is nonempty. We can do that as π is universally open. Then the fiber of a point $\mathrm{Spec}(k) \xrightarrow{p} \mathscr{U}$ must be a nonempty open subset of $\mathbb{A}^n_k$ for some n. Then by lemma (2.5) we know that U is must be a smooth-Nisnevich neighbourhood of p if k is infinite, and a smooth m-Nisnevich neighbourhood of p in general.

Let us now call P_v the element of $[X/G]\,(K)$ obtained by the generic point of $[(X \times U)/G]$. Suppose we have a smooth map $f : A \to [X/G]$ with a lifting of P_v. The lifting of P_v can be extended to a map i from an open subset $\mathscr{V}$ of $[(X \times U)/G]$ to A, making the following diagram commute:



Then all points that can be lifted to $\mathscr{V}$ can be lifted to A, and f is a smooth-Nisnevich cover of a nonempty open substack of $[X/G]$.

We can now use the finite type hypothesis to construct our finite covering family inductively. We start by taking a versal object P_1 for $\mathscr{M}$ and a map $p_1 : X_n \to \mathscr{M}$ such that P_1 lifts to p_1. Then p_1 must be a smooth-Nisnevich cover of an open subset $\mathscr{U}$. We then take versal objects for the finitely many irreducible components components of the complement of $\mathscr{U}$, and add to our collection enough maps to lift them all. Then we have obtained a smooth-Nisnevich cover (resp. smooth m-Nisnevich if we are working on finite fields) for an open subset whose complement has codimension at least 2. As our stack is Noetherian, this process will eventually end, giving us a smooth-Nisnevich (resp. smooth m-Nisnevich) cover of finite type of $\mathscr{M}$.

□

Given an algebraic stack $\mathscr{M}$ we denote *AlStk*$/\mathscr{M}$ the 2-category consisting of representable maps of algebraic stacks $\mathscr{N} \to \mathscr{M}$ with morphisms given by 2-commutative squares over the identity of $\mathscr{M}$. As we are requiring all maps to be representable, it is equivalent to a 1-category.

We define the (very big) smooth-Nisnevich site $(\mathit{AlStk}/\mathscr{M})_{\text{sm-Nis}}$ by allowing all smooth-Nisnevich maps as covers.

We define the (very big) arithmetic smooth m-Nisnevich site $(\mathit{AlStk}/\mathscr{M})_{\text{sm m-Nis}}$ by allowing all arithmetic smooth-Nisnevich maps as covers.

Lemma 2.7. *Let K be a field and let $\mathcal{U} \to \operatorname{Spec}(K)$ be a m-Nisnevich cover of* $\operatorname{Spec}(K)$*. Then we have:*

$$\check{H}^0(\mathcal{U}, \mathrm{H}^\bullet) = \mathrm{H}^\bullet(\operatorname{Spec}(K))$$

That is, the functor $\mathrm{H}^\bullet$ satisfies the sheaf condition with respect to m-Nisnevich covers of spectra fo fields.

Proof. Fix a m-Nisnevich cover $U \to \operatorname{Spec}(K)$. We can restrict to a finite cover. It is going to be in the form

$$\operatorname{Spec}(K_1) \quad \dots \quad \operatorname{Spec}(K_r) \to \operatorname{Spec}(K)$$

where $K_1, \dots K_r$ are finite separable extensions of K and

$$([K_1 : K], \dots, [K_r : K], m) := (d_1, \dots, d_r, m) = 1$$

Recall that given any scheme Y étale over the spectrum of a field K' there is a transfer map $t : \mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(\mathrm{Spec}(K'))$ given by taking for every point $\mathrm{Spec}(E) \to Y$ the norm map $\mathrm{N}^E_{K'}$. This is described in [Ros96, sec.1, 1.11]

Fix $a_1, \dots, a_r$ such that $a_1 d_1 + \dots + a_r d_r \equiv 1 (\mathrm{mod}\, m)$. For any scheme Y étale over $\mathrm{Spec}(K)$ we define a transfer map $T : \mathrm{H}^\bullet(Y \times_K U) \to \mathrm{H}^\bullet(Y)$ by taking for each K_i the usual transfer map $t : \mathrm{H}^\bullet(Y \times_K \mathrm{Spec}(K_i)) \to \mathrm{H}^\bullet(Y)$ point by point and multiplying it by a_i. Using the properties of the norm map it is immediate to check that T is a retraction for the pullback $\mathrm{H}^\bullet(Y) \to \mathrm{H}^\bullet(Y \times_K U)$.

For a positive integer s, let $\mathcal{H}^i$ be the sheafification in the m-Nisnevich topology of the presheaf $X \to H^i_{\acute{e}t}(X, \mu_p^s)$. There is a Čech to cohomology spectral sequence

$$E_2^{ij} = \check{H}^i(\mathcal{U}, \mathcal{H}^i) \Rightarrow H^{i+j}_{\acute{e}t}(\mathrm{Spec}(K), \mu_p^s)$$

coming from the covering $U \to \mathrm{Spec}(K)$. Our claim is equivalent to saying that for all s we have $H^s_{\acute{e}t}(\mathrm{Spec}(K), \mu_p^s) = \check{H}^0(\mathcal{U}, \mathcal{H}^s)$. By the spectral sequence above to do so it suffices to prove that $\check{H}^j(\mathcal{U}, \mathcal{H}^r) = 0$ for all $j > 0, r \geq 0$.

Let $\mathcal{U}'$ be the pullback to U of $\mathcal{U}$. As $\mathcal{U}'$ admits a section, the Čech cohomology groups $\check{H}^j(\mathcal{U}', \mathcal{H}^r)$ are zero for $j > 0$. There is a natural pullback map of Čech complexes between the complex of $\mathcal{U}$ and $\mathcal{U}'$, and the transfer map T defines a retraction of this map. Then this implies that the Čech cohomology groups $\check{H}^j(\mathcal{U}, \mathcal{H}^r)$ must be zero too for $j > 0$, proving our claim. □

Theorem 2.8. $\mathrm{Inv}^\bullet$ *is a sheaf in the smooth-Nisnevich topology. Let m be a non negative integer divisible by p. Then* $\mathrm{Inv}^\bullet$ *is a sheaf in the smooth m-Nisnevich topology.*

Proof. We begin with the smooth-Nisnevich case. First, notice that as the cohomological invariants of $\mathrm{Spec}(k)$ are equal to its cohomology, if α is a cohomological invariant of $\mathscr{M}$ and $p :\in \mathscr{M}(k)$ a point the pullback $p^*(\alpha)$ is the value of α at p.

Now, let $f : \mathscr{M} \to \mathscr{N}$ be a Nisnevich cover, and α a cohomological invariant of $\mathscr{M}$ satisfying the gluing condition. Let q be a point of $\mathscr{N}$ and $p, p' : \mathrm{Spec}(k) \to \mathscr{M}$ two different liftings of q. By the gluing conditions,

$$\alpha(p) = \mathrm{Pr_1}^*(\alpha)(p \times_{\mathscr{N}} p') = \mathrm{Pr_2}^*(\alpha)(p \times_{\mathscr{N}} p') = \alpha(p').$$

We may thus define a candidate invariant β by $\beta(q) = \alpha(p)$, where p is any lifting of q.

It is clear that β is a natural transformation between the functor of points of $\mathscr{N}$ and $\mathrm{H}^\bullet(-)$. We need to prove it has property (1.1).

Let R be a Henselian DVR, $i : \mathrm{Spec}(R) \to \mathscr{N}$ a morphism. The morphism $\mathrm{Pr}_2 : \mathscr{M} \times_{\mathscr{N}} \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ is a Nisnevich cover of the spectrum of an Henselian ring, so it has a section. This section provides a map $\mathrm{Spec}(R) \to \mathscr{M}$. By evaluating α at the image of the generic and closed point of $\mathrm{Spec}(R)$, we obtain the desired result.

The general statement follows from the reasoning above and lemma 2.7. For the last part we only need to notice that if $U \to \mathrm{Spec}(K)$ is an m-Nisnevich cover and R is a Henselian K-algebra then the induced pullback map

$$\mathrm{H}^\bullet(\mathrm{Spec}(R)) \to \mathrm{H}^\bullet(\mathrm{Spec}(R) \times_K U)$$

is injective. □

We used such a big category to get the strongest statement and also to have a category with the final object $\mathrm{Id} : \mathscr{M} \to \mathscr{M}$ as a term of comparison. With the next proposition we see that we can reduce our scope to tamer sites.

Definition 2.9. Denote $Spc/\mathscr{M}$ the category of $\mathscr{M}$-algebraic spaces, with morphisms cartesian squares over the identity of $\mathscr{M}$. Denote $Sm/\mathscr{M}$ the full subcategory of $Spc/\mathscr{M}$ consisting of algebraic spaces smooth over k_0. On these two categories we consider the Nisnevich sites $(Spc/\mathscr{M})_{\mathrm{Nis}}$ and $(Sm/\mathscr{M})_{\mathrm{Nis}}$ where the coverings are étale Nisnevich maps, and the smooth-Nisnevich sites $(Spc/\mathscr{M})_{\text{sm-Nis}}$ and $(Spc/\mathscr{M})_{\text{sm-Nis}}$ where the covers are smooth-Nisnevich maps.

For each of these sites we define a corresponding m-Nisnevich site $(Spc/\mathscr{M})_{\text{m-Nis}}$, $(Sm/\mathscr{M})_{\text{m-Nis}}$, $(Spc/\mathscr{M})_{\text{sm m-Nis}}$, $(Spc/\mathscr{M})_{\text{sm m-Nis}}$

The site $(Sm/\mathscr{M})_{\mathrm{Nis}}$ could be called "Lisse-Nisnevich" site in analogy with the usual Lisse-étale site on algebraic stacks, and in fact it is a subsite of $\mathscr{M}_{\text{Lis-ét}}$. We have also defined the big sites in analogy with the approach used in [Sta15, 06TI] as these are the one we are working with in most of the proofs. the next corollary shows that we can work indifferently in each of these sites.

Corollary 2.10. *The $(AlStk/\mathscr{M})_{\text{sm-Nis}}$ (resp. $(AlStk/\mathscr{M})_{\text{sm m-Nis}}$) site and the sites defined in (2.9) all induce the same topos.*

Proof. This is a consequence of propositions (2.6,2.3) and the chains of inclusions

$$(Sm/\mathscr{M})_{\text{Nis}} \subseteq (Sm/\mathscr{M})_{\text{sm-Nis}} \subseteq (AlgSt/\mathscr{M})_{\text{sm-Nis}}$$

$$(Spc/\mathscr{M})_{\text{Nis}} \subseteq (Spc/\mathscr{M})_{\text{sm-Nis}} \subseteq (AlgSt/\mathscr{M})_{\text{sm-Nis}}$$

The same works word by word for the m-Nisnevich sites. □

This gives us the tautologic equality $\text{Inv}^\bullet(\mathscr{M}) = H^0((Sm/\mathscr{M})_{\text{Nis}}, \text{Inv}^\bullet)$. In the next section we will use this equality and the fact that $(Sm/\mathscr{M})_{\text{Nis}}$ is a site of smooth algebraic spaces to obtain a satisfactory description of the sheaf $\text{Inv}^\bullet$.

3 $\text{Inv}^\bullet$ *as a derived functor*

In this section we give an explicit description of the sheaf of cohomological invariants as the sheafification of the étale cohomology with respect to the smooth-Nisnevich site. This will immediately give us a clear idea on how our invariants should be computed and their properties.

To keep the statements short, we will work on the ordinary Nisnevich and smooth-Nisnevich sites. We can do this without loss of generality as the results for the m-Nisnevich sites will be obtained for free from the ordinary case.

Definition 3.1. Let $\mathscr{M}$ be an algebraic stack, and let $i : (Sm/\mathscr{M})_{\text{Nis}} \to (Sm/\mathscr{M})_{\text{ét}}$ the inclusion of $(Sm/\mathscr{M})_{\text{Nis}}$ in the Lisse-étale site of $\mathscr{M}$. It induces a left-exact functor i_* from the Lisse-étale topos of $\mathscr{M}$ to the topos of $(Sm/\mathscr{M})_{\text{Nis}}$.

We will call $\text{RInv}^\bullet := \oplus_j \, \text{R}^j \, i_*(\mu_p^{\otimes j})$ the sheaf of *regular invariants.*

We can see the sheaf of regular invariants as the sheafification of the presheaf $U \to \text{H}^\bullet(U)$ in any of the sites defined in the previous section.

Remark 3.2. If R is an Henselian ring then $\text{RInv}^\bullet(\text{Spec}(R))$ is naturally isomorphic to $\text{H}^\bullet(\text{Spec}(R))$.

The map from étale cohomology to cohomological invariants naturally extends to a map of sheaves between regular invariants and cohomological invariants. The previous remark shows that this map can be again interpreted as sending an element $\alpha \in \text{RInv}^\bullet(\mathscr{M})$ to the cohomological invariant $\tilde{\alpha} \in \text{Inv}^\bullet(\mathscr{M})$ defined by sending a point $p \in \mathscr{M}(K)$ to $\tilde{\alpha}(p) = p^*(\alpha)$.

Proposition 3.3. *The map* $\tilde{*} : \mathrm{RInv}^\bullet \to \mathrm{Inv}^\bullet$ *is injective.*

Proof. Suppose a given regular invariant α is zero as a cohomological invariant. By lemma (1.1), the pullback of a regular invariant to the spectrum of an Henselian local ring is the same as the pullback to its closed point. The fact that α is zero as a cohomological invariant then implies that the pullback of alpha to the spectrum of any local Henselian ring is zero, as it is zero at its closed point. Then α must be zero, as regular invariants form a sheaf in the Nisnevich topology. □

This shows that we can think of $\mathrm{RInv}^\bullet$ as a subsheaf of $\mathrm{Inv}^\bullet$. We want to prove the following:

Theorem 3.4. *Let* $\mathscr{M}$ *be an algebraic stack smooth over* k_0. *Then* $\mathrm{RInv}^\bullet(\mathscr{M}) = \mathrm{Inv}(\mathscr{M})$.

We will use a few lemmas. First we prove that for a smooth connected space a cohomological invariant is determined by its value at the generic point.

Lemma 3.5. *Let* R *be a regular Henselian local* k_0*-algebra, with residue field* k *and quotient field* K. *Let* α *be a cohomological invariant of* $\mathrm{Spec}(R)$. *Then if* $\alpha(\mathrm{Spec}(K)) = 0$ *we have* $\alpha(\mathrm{Spec}(k)) = 0$.

Proof. We will proceed by induction on the dimension d of R. The case $d = 0$ is trivial, and the case $d = 1$ is proven in [GMS03, 7.7]. Suppose now $d > 1$.

Let x be a non invertible, nonzero element of R, and $R_1 = R/(x)$. R_1 is Henselian, and by the inductive hypothesis we know that if the value of α at $\mathrm{Spec}(k(R_1))$ is zero then the value of α at $\mathrm{Spec}(k)$ must be zero too.

Let now $x, \ldots, x_{d-1}$ be a regular sequence for R. Consider $R_2 := (R_{x_1 \ldots x_{d-1}})^h$. The residue field of R_2 is $k(R_1)$, and its quotient field is $k(R)$.

Let R_2^h be the Henselization of R_2, and consider the pullback α' of α through the map $\mathrm{Spec}(R_2^h) \to \mathrm{Spec}(R)$. We have $\alpha'(\mathrm{Spec}(k(R_1)) = \alpha(\mathrm{Spec}(k(R_1)))$, and $\alpha(\mathrm{Spec}(K) = 0$ implies the same for the generic point of $\mathrm{Spec}(R_2^h)$. Then $\alpha(\mathrm{Spec}(k(R)) = 0$ implies $\alpha(\mathrm{Spec}(k(R_1))) = 0$ which in turn implies $\alpha(\mathrm{Spec}(k)) = 0$. □

Example 3.6. This fails as soon as X is no longer normal. Let $R = \{\phi \in \mathbb{C}[[t]] \mid \phi(0) \in \mathbb{R}\}$. R is an Henselian ring of dimension one, with residue field $\mathbb{R}$ and quotient field $\mathbb{C}[[t]]$, but $\mathrm{H}^1(\mathrm{Spec}(\mathbb{R}), \mathbb{F}_2) \neq 0$, while $\mathrm{H}^1(\mathrm{Spec}(\mathbb{C}[[t]]), \mathbb{F}_2) = 0$.

Corollary 3.7. *Let* X *be an irreducible scheme smooth over* k_0. *A cohomological invariant* α *of* X *is zero if and only if its value at the generic point of* X *is zero.*

Proof. Let α be a cohomological invariant of X such that its restriction at the generic point μ is zero. Le p be another point, and let R be the local ring of p in the smooth-Nisnevich topology, μ_1 the its generic point. As μ_1 is obtained by base change from μ, $\alpha(\mu_1)$ must be zero. Then, by the previous lemma, $\alpha(p)$ is zero. □

The same happens for regular invariants.

Lemma 3.8. *Let X be a scheme smooth over k_0. Let $\mathscr{H}^\bullet$ be the sheafification of the étale cohomology in the Zariski topology. There is a natural isomorphism of Zariski sheaves $\mathscr{H}^\bullet \simeq \mathrm{H}^\bullet_{nr}$ given by restriction to the generic point.*

Proof. This is proven by the Gersten resolution [BO74, 4.2.2]. □

By remark 1.4 we know that the value of a cohomological invariant α at the generic point of a smooth space X belongs to the unramified cohomology $H_{nr}(X)$. We only have to put together the lemmas previous lemmas to conclude.

Proposition 3.9. *Let X be a scheme smooth over k_0. There is a natural isomorphism $\mathrm{Inv}^\bullet(X) \simeq \mathrm{H}^\bullet_{nr}(X)$. In particular, all invariant of X are regular.*

Proof. We will prove the proposition for an irreducible smooth scheme. The general statement follows. Consider these three morphisms:

- The map $\tilde{*} : \mathscr{H}(X) \to \mathrm{Inv}^\bullet(X)$ given by restricting to points.
- The map $\mathrm{res}_1 : \mathscr{H}(X) \to \mathrm{H}^\bullet_{nr}(X)$ given by restricting to the generic point.
- The map $\mathrm{res}_2 : \mathrm{Inv}(X) \to \mathrm{H}^\bullet_{nr}(X)$ given by evaluating at the generic point.

The second map is an isomorphism by the previous lemma, and the third map is injective by Corollary 3.7. As clearly $\mathrm{res}_2 \circ \tilde{*} = \mathrm{res}_1$, the three maps must all be isomorphisms.

As $\mathrm{RInv}^\bullet$ is the Nisnevich sheafification of $U \to \mathscr{H}(U)$ the result follows. □

Remark 3.10. Proposition 3.9 implies that given a regular Henselian ring R, with closed and generic points respectively p, P the equation $\alpha(P) = j(\alpha(P))$, as in (1.1), holds for any cohomological invariant α of $\mathrm{Spec}(R)$. This shows that in the definition of cohomological invariant we could equivalently choose to require the (apparently) stronger property that equation (1.1) held for all regular Henselian rings, rather than just for DVRs.

Proof of Theorem 3.4. We can just plug the previous results in the tautological equality $\mathrm{Inv}^\bullet(\mathscr{M}) = H^0((Sm/\mathscr{M})_{\mathrm{Nis}}, \mathrm{Inv}^\bullet)$ obtaining

$$\mathrm{Inv}^\bullet(\mathscr{M}) = H^0((Sm/\mathscr{M})_{\mathrm{Nis}}, \mathrm{Rinv}^\bullet) = H^0((Sm/\mathscr{M})_{\mathrm{Nis}}, (\mathrm{H}^\bullet)^{\mathrm{Nis}})$$

Where $(\mathrm{H}^\bullet)^{\mathrm{Nis}}$ denotes that we are taking the sheafification in the Nisnevich topology. Then by the standard description of derived functors we get

$$\mathrm{Inv}^j(\mathscr{M}) = \mathrm{R}^j\, i_*(\mathbb{F}_p(j))(\mathscr{M})$$

□

Corollary 3.11. *The same results hold for the m-Nisnevich sites if and only if p divides m.*

Proof. This is clear as we already know that that cohomological ivnariants form a sheaf in the finer m-Nisnevich topologies. □

We can use the description of the cohomological invariants on schemes to deduce two important properties of cohomological invariants.

Lemma 3.12.

- *Let $U \to X$ be an open immersion of schemes, such that the codimension of the complement of N is at least 2. Then $\mathrm{H}^\bullet_{nr}(X) = \mathrm{H}^\bullet_{nr}(U)$*
- *An affine bundle $E \to X$ induces an isomorphism on unramified cohomology.*

Proof. The first statement is true by definition and the second is proven in [Ros96, 8.6]. □

Proposition 3.13. *Let $\mathscr{N} \to \mathscr{M}$ be an open immersion of algebraic stacks, such that the codimension of the complement of $\mathscr{N}$ is at least 2. Then $\mathrm{Inv}^\bullet(\mathscr{M}) = \mathrm{Inv}^\bullet(\mathscr{N})$.*

Proof. Let $\pi : X \to \mathscr{M}$ be an element of smooth-Nisnevich cover of $\mathscr{M}$ by a scheme. As all the elements we will consider belong to *AlStk*$/\mathscr{M}$ we write $A \times B$ for $A \times_{\mathscr{M}} B$. Name U the open subscheme $X \times \mathscr{N}$ of X. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Inv}^\bullet(\mathscr{M}) & \xrightarrow{\pi^*} & \mathrm{Inv}^\bullet(X) & \overset{\mathrm{Pr_1}^*}{\underset{\mathrm{Pr_2}^*}{\rightrightarrows}} & \mathrm{Inv}^\bullet(X \times X) \\
\downarrow i^* & & \downarrow i_1^* & & \downarrow i_2{}^* \\
\mathrm{Inv}^\bullet(\mathscr{N}) & \xrightarrow{\pi_1^*} & \mathrm{Inv}^\bullet(U) & \overset{\mathrm{Pr_1}^*}{\underset{\mathrm{Pr_2}^*}{\rightrightarrows}} & \mathrm{Inv}^\bullet(U \times_{\mathscr{N}} U)
\end{array}$$

As $i_1{}^*, i_2{}^*$ are isomorphisms (a smooth map fixes codimension), the elements of $\mathrm{Inv}^\bullet(X)$ satisfying the gluing conditions are the same as those of $\mathrm{Inv}^\bullet(U)$. □

Proposition 3.14. *Let $\mathscr{M}$ be an algebraic stack smooth over k_0. An affine bundle $\rho : \mathscr{V} \to \mathscr{M}$ induces an isomorphism on cohomological invariants.*

Proof. Consider a smooth-Nisnevich cover $f : X \to \mathscr{M}$. We have a cartesian square

$$\begin{array}{ccc} \mathscr{V} \times_{\mathscr{M}} X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle f} \\ \mathscr{V} & \xrightarrow{\rho} & \mathscr{M} \end{array}$$

The horizontal arrows are affine bundles, and the vertical arrows are smooth-Nisnevich covers. Moreover, we can choose f to trivialize $\mathscr{V}$. The rings of cohomological invariants of X and $\mathscr{V} \times_{\mathscr{M}} X$ are isomorphic, and we can easily see that the gluing conditions hold for an invariant of $\mathscr{V} \times_{\mathscr{M}} X$ if and only if they hold for the corresponding invariant of X. □

By putting together these results we get an alternative proof of Totaro's theorem from [GMS03, appendix C]:

Theorem 3.15 (Totaro). *Let G be an affine algebraic group smooth over k_0. Suppose that we have a representation V of G and a closed subset $Z \subset V$ such that the codimension of Z in V is 2 or more, and the complement $U = V \setminus Z$ is a G-torsor. Then the group of cohomological invariants of G is isomorphic to the unramified cohomology of U/G.*

Proof. The map $[V/G] \to \mathrm{B}\,G$ is a vector bundle, so by proposition 3.14 it induces an isomorphism on cohomological invariants by pullback. As $U/G \to [V/G]$ is an open immersion satisfying the requirements of proposition 3.13, it induces an isomorphism on cohomological invariants too. □

4 The invariants of $\mathscr{M}_{1,1}$

As a first application of the results in this section, we compute the cohomological invariants of the stack $\mathscr{M}_{1,1}$ of elliptic curves. The computation will use Rost's Chow groups with coefficients, but it will not require any of the additional techniques we will develop in the following sections.

Recall that by [Ros96, 6.5] for a smooth scheme X we have $\mathrm{Inv}^\bullet(X) = \mathscr{H}^\bullet(X) = A^0(X; \mathrm{H}^\bullet)$, where $A^0(X; \mathrm{H}^\bullet)$ is the 0-codimensional Chow group with coefficients, and the coefficients functor is étale cohomology. In the proof we will shorten $A^\bullet(X; \mathrm{H}^\bullet)$ to $A^\bullet(X)$.

Also recall that an algebraic group G is called *special* if any G-torsor is Zariski-locally trivial. This implies that for any X being acted upon by G, the map $X \to [X/G]$ is a smooth-Nisnevich cover. Examples of special groups are GL_n and Sp_n.

Theorem 4.1. *Suppose the characteristic of k_0 is different from two or three. Then the cohomological invariants of $\mathscr{M}_{1,1}$ are trivial if $(p, 6) = 1$.*

Otherwise,

$$\mathrm{Inv}^\bullet(\mathscr{M}_{1,1}) \simeq \mathrm{H}^\bullet(k_0)\,[t] \big/ (t^2 - \{-1\}t)$$

where the degree of t is 1.

The generator t sends an elliptic curve over a field k with Weierstrass form $y^2 = x^3 + ax + b$ to the element $[4a^3 + 27b^2] \in k^*/k^{*p} \simeq \mathrm{H}^1(\mathrm{Spec}(k))$.

Proof. Recall that if the characteristic of k_0 is different from 2 and 3 we have $\mathscr{M}_{1,1} \simeq [X/G_m]$, where $X := \mathbb{A}^2 \setminus \{4x^3 = 27Y^2\}$ and the action of G_m is given by $(x, y, t) \to (xt^4, yt^6)$.

We will first determine the invariants of X. As the multiplicative group is special, $X \to \mathscr{M}_{1,1}$ is a smooth-Nisnevich cover, so after we compute $\mathrm{Inv}^\bullet(X)$ all we have to do is check the gluing conditions.

Consider now the closed immersion $G_m \to \mathbb{A}^2 \setminus (0,0)$ induced by the obvious map $\mathbb{G}_m \to \{4x^3 = 27Y^2\} \setminus (0,0)$ given by the normalization $\mathbb{A}^1 \to \{4x^3 = 27Y^2\}$. We have an exact sequence [Ros96, sec.5]

$$0 \to \mathrm{A}^0(\mathbb{A}^2 \setminus (0,0)) \to \mathrm{A}^0(X) \xrightarrow{\partial} \mathrm{A}^0(\mathbb{G}_m) \to \mathrm{A}^1(\mathbb{A}^2 \setminus (0,0)) \to \mathrm{A}^1(X) \xrightarrow{\partial} \mathrm{A}^1(\mathbb{G}_m)$$

To compute $\mathrm{A}^\bullet(\mathbb{A}^2 \setminus (0,0))$, we use a second exact sequence:

$$\mathrm{A}^1(\mathbb{A}^2) \to \mathrm{A}^1(\mathbb{A}^2 \setminus (0,0)) \to \mathrm{A}^0(\mathrm{Spec}(k)) \to \mathrm{A}^2(\mathbb{A}^2)$$

As $\mathbb{A}^2$ is a vector bundle over a point, the first and last term are zero, implying that $\mathrm{A}^1(\mathbb{A}^2 \setminus (0,0))$ is generated as a $\mathrm{A}^\bullet(\mathrm{Spec}(k_0))$-module by a single element in degree one. By the results in the previous section, we have $\mathrm{A}^0(\mathbb{A}^2 \setminus (0,0)) = \mathrm{A}^0(\mathbb{A}^2) = \mathrm{A}^*(\mathrm{Spec}(k_0))$. As $\mathrm{A}^2(\mathbb{A}^2)$ is zero, the continuation to the left of the sequence above implies that also $\mathrm{A}^2(\mathbb{A}^2 \setminus (0,0))$ is zero.

Using the same technique we find that $\mathrm{A}^0(\mathbb{G}_m, \mathrm{H}^\bullet)$ is a free $\mathrm{H}^\bullet(\mathrm{Spec}(k_0)$-module generated by the identity in degree zero and an element t in degree one.

We can now go back to the first exact sequence. Using all the data we obtained, we find that $\mathrm{A}^0(X)$ is generated as a free $\mathrm{A}^\bullet(\mathrm{Spec}(k_0))$ module by the identity and an element α in degree one.

Finding out what α is turns out to be easy, as $\mathrm{H}^1(X)$ is generated by $[4x^3 + 27y^2]$, seen as al element of $\mathscr{O}_X^*/\mathscr{O}_X^{*p}$, and this is clearly nonzero as a cohomological invariant.

The last thing we need to do is to check the gluing conditions; let $m : X \times_{\mathscr{M}_{1,1}} X = X \times \mathbb{G}_m \to X$ be the multiplication map, and let $\pi : X \times \mathbb{G}_m \to X$ be the first projection. Consider the points $q, q' : \mathrm{Spec}(k(x, y, t)) \to X$ defined respectively by $x \to x, y \to y$ and $x \to t^4x, y \to t^6y$. The values of α at q and q' are respectively equal to $m^*(\alpha)(\mu)$ and $\pi^*(\alpha)(\mu)$, where μ is the generic point of $X \times \mathbb{G}_m$. It is necessary and sufficient for α to verify the gluing conditions that $\alpha(q) = \alpha(q')$.

We have $\alpha(q) = [4x^3 + 27y^2]$, $\alpha(q') = [t^{12}(4x^3 + 27y^2)]$. These two elements of $\mathrm{H}^1(\mathrm{Spec}(k(x, y, t))$ are clearly equal if and only if p divides 12.

Finally, the relation $\alpha^2 = \{-1\}\alpha$ is due to the fact that when we identify $\mathrm{H}^1(k)$ with $k^*/(k^*)^p$ we have $\{a\}^2 = \{-1\}\{a\}$ for any $a \in k^*$. One can see this as a consequence of the relations in Milnor's K-theory, as the morphism from Milnor's K-theory of k to $\mathrm{H}^\bullet(k)$ is surjective in degree 1, see [Ros96, pag.327 and rmk. 1.11]. □

5 *Generalized cohomological invariants*

Our aim in this section is to construct a smooth-Nisnevich sheaf of cohomological invariants with values in a given cycle module M, satisfying the same description we have when M is equal to Galois cohomology with torsion coefficients.

There are two main problems to be fixed here: first, we must find a new continuity condition, and secondly, we need to find a different way to identify the cohomological invariants of a smooth scheme X with $A^0(X, M)$, as we do not have étale cohomology to act as a medium between the two.

In the following we will often restrict to only considering *geometric* discrete valuation rings. This means that the ring R is a k_0 algebra and the transcendence degree over k_0 of its residue field is equal to that of its quotient field minus one. This is the same as asking that our DVR is the local ring of an irreducible variety at a regular point of codimension one.

Lemma 5.1. *Let R be a DVR with valuation v, with quotient field F and residue* *field k. Let M be a cycle module. We denote by $K(v)$ the ring $K(F)/(1+m_v)$, where $K(F)$ is Milnor's K-theory of F, and by $M(v)$ the $K(v)$-module $M(k) \otimes_{K(k)} K(v)$.*

There are maps $i : M(k(v)) \to M(v)$ and $p : M(k(R)) \to M(v)$, and a split exact sequence:

$$0 \to M(k(v)) \xrightarrow{i} M(v) \xrightarrow{\partial} M(k(v)) \to 0$$

Proof. This is [Ros96, rmk. 1.6]. □

one easily sees that when M is equal to Galois cohomology and R is Henselian then $M(F) = M(v)$ and the continuity condition for a cohomological invariant is equivalent to asking that $p(F) = i(\alpha(k)$. This motivates the following definition:

Definition 5.2. Let $\mathscr{X}$ be an algebraic stack, and M a cycle module. A cohomological invariant for $\mathscr{X}$ with coefficients in M is a natural transformation between the functor of points

$$F_{\mathscr{X}} : (\text{field}/k_0) \to (\text{set})$$

And M, satisfying the following property: given a Henselian DVR R as above and a map $\mathrm{Spec}(R) \to \mathscr{X}$ we have $p(F) = i(\alpha(k)$ in $M(v)$. We denote the functor of cohomological invariants with coefficients in M by $\mathrm{Inv}^{\bullet}(-, M)$.

Remark 5.3. The condition above is equivalent to asking that for any irreducible normal scheme $X \to \mathscr{X}$ the value of α at $k(X)$ belongs to $A^0(X)$, and that for a Henselian DVR R we have $\alpha(k(v)) = s_v(\alpha(k(R)))$, where s_v is the map defined in [Ros96, sec.1, D4].

Theorem 5.4. *Cohomological invariants with coefficients in M form a sheaf in the smooth-Nisnevich topology and smooth 0-Nisnevich topology.*

Proof. We can just repeat word by word the reasoning in (5.4). □

Lemma 5.5. *When X is a scheme smooth over k_0 we have an injective map $A^0(X, M) \to \mathrm{Inv}(X, M)$ assigning to an element $\alpha \in A^0(X, M)$ the invariant $\tilde{\alpha}$ defined by $\tilde{\alpha}(p) = p^*(\alpha)$.*

Proof. The only thing to check here is that the continuity condition is satisfied.

First one should note that by blowing up the image of the closed point we can restrict to geometric DVRs. Given a geometric DVR (R, v), we can see it as the local ring of a regular variety at a point of codimension one. Then by [Ros96, 11.2] we know that the specialization map $r \circ J(i)$ induced by the inclusion i of the closed point is the same as the map s_v.

It is immediate to verify that the pullback $A^0(X, M) \xrightarrow{f^*} A^0(\mathrm{Spec}(R), M)$ is compatible with the specialization $A^0(\mathrm{Spec}(R), M) \to A^0(\mathrm{Spec}(k(v)), M)$, so that we have $r \circ J(i) \circ f^* = (f \circ i)^*$ and we can conclude. □

Theorem 5.6. *Let $\mathscr{X}$ be an algebraic stack smooth over k_0. Then the sheaf of cohomological invariants of $\mathscr{X}$ with coefficients in a cycle module M is isomorphic to the functor $X \to A^0(X, M)$ in the smooth-Nisnevich (resp smooth 0-Nisnevich) site of $\mathscr{X}$.*

Proof. We can reason as in theorem (3.4). The map $A^0(X, M) \to \mathrm{Inv}(X, M)$ has an evident inverse given by taking the value at the generic point. □

2. REVISITING CHOW GROUPS WITH COEFFICIENTS

In this chapter we completely rework the theory of Chow groups with coefficients in the setting of algebraic spaces.

The first four sections correspond respectively to sections 1-2, 3 to 5, 7 to 9 and 10 to 14 of Rost's original paper [Ros96]. The reworking of Section 6 of the paper has been moved to the fourth section.

The last section defines a theory of projective bundles and Chern classes, and then treats the equivariant version of the theory, using the result at the beginning of the section to compute the equivariant Chow groups with coefficients of a point for some special groups.

1 Basic definitions

In this chapter all algebraic spaces are intended to be of finite type and quasi-separated over a fixed base field k_0

A cycle premodule M over k_0, as defined in [Ros96, sec 1], is a functor from the category of field extensions of k_0 to the category of (graded) abelian groups satisfying a long list of properties.

Before briefly reviewing what these properties state, we begin with a change in notation: rather then considering a covariant functor from a category of rings, we will think of a cycle module as a contravariant functor from the opposite category, that is, the category of extensions $\mathrm{Spec}(E) \to \mathrm{Spec}(k_0)$. While this choice is not the most natural, we will avoid having to switch upper and lower stars when passing from fields to schemes, which in the author's opinion can be a cause for considerable confusion.

That said, the properties and rules in [Ros96, sec 1] basically state that:

- For any field k, the group $M(k)$ is a left graded $K^\bullet(k)$-module, where $K^\bullet(k)$ is Milnor's K-theory ring of k, and this association has good functorial properties.
- There is a corestriction map $M(F) \xrightarrow{\phi_*} M(E)$ for a finite extension $\mathrm{Spec}(F) \xrightarrow{\phi}$

Spec(E), and it follows the usual properties of corestrictions, e.g. $\phi_* \circ \phi^* = [F : E] \operatorname{Id}_{M(E)}$, where $\phi^* = M(\phi)$.

- There is a graded-commutative pairing $\cdot : M \times M \to M$ of left graded $K^\bullet$-modules which is functorial and satisfies the projection formula $\phi_*(\phi^*(x) \cdot y) = x \cdot \phi_*(y)$.
- For a discrete valuation v on a field E we have a map of graded $K^\bullet$-modules $\delta_v : M(\operatorname{Spec}(E)) \to M(\operatorname{Spec}(k(v)))$ of degree -1 which is compatible with restriction, corestriction and the pairing.

For a normal scheme X, with generic point ξ_X, and a point $x \in X^1$ we can define a map $\delta_x : M(\xi_X) \to M(x)$ given by $\delta_{v(x)}$, where $v(x)$ is the valuation induced by $\mathscr{O}_{X,x} \to k_0(\xi_X)$.

For a general scheme X, and points x, y we can now define a map $\delta^y_x : M(x) \to M(y)$ by considering $Z = \overline{\{y\}}$. If $x \notin Z^1$ then $\delta^y_x = 0$. Otherwise, let $\tilde{Z} \xrightarrow{\pi} Z$ be a normalization of Z, and put

$$\delta^y_x = \sum_{\pi(w)=x} \pi_* \delta_w.$$

We can finally state what a cycle module is.

Definition 1.1. A cycle module is a cycle premodule satisfying the following two conditions:

1. (Finite support of divisors) Let X be a normal scheme and $\rho \in M(\xi_X)$. Then $\delta_x(\rho)$ is nonzero only for finitely many $x \in X^1$.
2. (Closedness) Let X be integral and local of dimension 2. Let x be its unique closed point. Then

$$\sum_{y \in X^1} \delta^y_x \circ \delta_y = 0$$

These properties are sufficient for a cycle module to work well for schemes, but it's not clear *a priori* if they suffice for a theory of Chow groups with coefficients for (quasi-separated) algebraic spaces. It's not even clear what one one should make of the maps δ^y_x, as for $x \in \overline{\{y\}}^{(1)}$ we do not necessarily have a meaningful valuation to consider.

Recall first that for a quasi separated algebraic space with X given a point x there is always a (unique) map $\operatorname{Spec}(K) \xrightarrow{\phi_x} X$ such that any other map $\operatorname{Spec}(E) \to X$ factors through ϕ [Sta15, 06HN].

It makes sense then to define a *Nisnevich neighbourhood* of x to be an étale map $Y \to X$ with a lifting of ϕ_x. It is proven in [Knu71, Ch.2,6.4] that given a point $x \in X$ there is always a Nisnevich neighbourhood of x such that Y is a scheme.

To extend the theory of Chow groups with coefficients to algebraic spaces, we want to compute the the "differential" maps δ_x^y, rather than in a Zariski neighbourhood of x, in a Nisnevich neighbourhood of x. The first thing to check is that the definition works for schemes:

Lemma 1.2. *Let X be a normal scheme, $x \in X^{(1)}$ a point and $U_x \xrightarrow{\phi_x} X$ a Nisnevich neighbourhood of x, with a lifting p of x. For a cycle module M, we identify the group $M(x)$ with $M(p)$ via the isomorphism $p \to x$. Let Then the map δ_x is equal to the composition of $M(\xi_X) \xrightarrow{\phi_x^*} M(\xi_{U_x})$ and δ_p.*

Proof. This is an immediate consequence of rule 3.a in [Ros96, Sec 1], which states that for a cycle module M, an extension $\mathrm{Spec}(F) \xrightarrow{\phi} \mathrm{Spec}(E)$ and a valuation v on F which restricts on E to a nontrivial valuation v' with ramification index e we have an equality $\delta_v \phi^* = e(\phi^* \delta_{v'})$ as maps from $M(E)$ to $M(k(v))$. In this case $e = 1$ and $\phi^* = \mathrm{Id}_{k(v)}$. □

This shows that for schemes we can equivalently take the differentials δ_x^y on any Nisnevich neighbourhood of x, leading to the following more general definition:

Definition 1.3. Let X be a normal algebraic space with quasi compact diagonal, $x \in X^1$. We define the map $\delta_x : M(\xi_X) \to M(x)$ by $\delta_x = \delta_p \phi^*$, where U is a scheme and $U \xrightarrow{\phi} X$ is any Nisnevich neighbourhood of x, the point p is any lifting of x and ϕ^* is the restriction map induced by $p \xrightarrow{\phi} x$.

For a general algebraic space X, and points x, y we define the map $\delta_x^y : M(x) \to M(y)$ by considering $Z = \{y\}$. If $x \notin Z^1$ then $\delta_x^y = 0$. Otherwise, let $\tilde{Z} \xrightarrow{\pi} Z$ be a normalization of Z, and put

$$\delta_x^y = \sum_{\pi(w)=x} \pi_* \delta_w$$

.

We now check that the two conditions of definition (1.1) transpose directly to the setting of algebraic spaces:

Proposition 1.4. *Let X be a normal algebraic space. Then:*

1. *The map δ_x is nonzero only for finitely many points $x \in X^{(1)}$.*

2. *Given a point $y \in X^{(2)}$, we have*

$$\sum_{y \in X^1} \delta_x^y \circ \delta_y = 0$$

Proof. 1. This is an immediate consequence of the fact that an algebraic space with quasi compact diagonal is generically a scheme and Noetherianity.

2. Let U be a scheme, $U \xrightarrow{\phi} X$ a Nisnevich neighbourhood of x, with a lifting p. Consider an element $\alpha \in M(\xi_X)$. For all $y \in X^{(1)}$ such that $x \in \overline{\{y\}}$ there is a unique point $y' \in U^{(1)}$ with $p \in \overline{\{y'\}}$ mapping to y. By the same reasoning as in the previous lemma, we see that $\phi^* \circ \delta_y(\alpha) = \delta^{y'} \circ \phi^*(\alpha)$ in $M(y')$. Now we can compare $\delta_x^y \circ \delta_y(\alpha)$ with $\delta_p^{y'} \circ \delta^{y'} \circ \phi^*(\alpha) = \delta_p^{y'} \circ \phi^* \circ \delta^y(\alpha)$; if we choose U as the Nisnevich neighbourhood for computing δ_x^y these two are clearly equal. This shows that

$$\sum_{y \in X^1} \delta_x^y \circ \delta_y = \sum_{y \in U^1} \delta_p^y \circ \delta_y$$

And the right-hand term is clearly zero due to the second point of the definition for schemes.

□

With this settled, the definition of the cycle groups C_p is immediate:

Definition 1.5. Let X be an algebraic space with. The Cycle complex of X with coefficients in M is defined as the pair

$$\left(\bigoplus_{p \geq 0} C_p, \mathrm{d}_X \right)$$

Where $C_p = \bigoplus_{x \in X_p} M(x)$ and d_X is the operator of degree -1 defined by $\mathrm{d}_X(\alpha) = \sum_{x,y \in X} \delta_x^y(\alpha)$. Points (1) and (2) of proposition (1.4) ensure respectively that d_X is well defined and that $\mathrm{d}_X \circ \mathrm{d}_X = 0$.

We will freely switch to the codimension notation $C^q = C_{\mathrm{Dim}(X)-q}$.

We define the Chow groups with coefficients of X as $A_i(X) = H_i(C_p)$, or equivalently, if we are using the codimension notation as $A^i(X) = H^i(C^p)$.

The Chow groups have two natural gradings induced by the grading of the cycle module M and by codimension. To avoid confusion, the term degree will always refer to the degree induced by this grading, and we will use the term codimension when referring to the grading in $A^\bullet(X)$ induced by codimension. The codimension is considered as an even grading, so that the total degree of an element $\alpha \in A^p(X, M(d))$ will be $2p+d$. As we will see, the ring $A^*(X)$ will be graded-commutative with respect to this first grading, and also operators between Chow groups with coefficients will satisfy the Koszul sign rule with respect to it.

We will occasionally write $A^n(X, d)$ to denote the subgroup of $A^n(X)$ given by elements of degree d.

2 Maps and compatibilities

In this subsection we will define the usual maps that one would expect in a theory of Chow groups, and show that they satisfy all the reasonable requests of compatibility. We will try to make it clear when equalities hold at cycle level rather than just at homology level, as this may turn out to be useful when a finer analysis is needed.

Definition 2.1. For a morphism $X \xrightarrow{f} Y$ of algebraic spaces we define:

1. A pushforward $f_* : C_p(X) \to C_p(Y)$. For a cycle α with support on a point $p \in X$, put $w = f(p)$. we define $f_*\alpha = 0$ if the extension $p \xrightarrow{f_{|p}} w$ is not finite, and $f_*\alpha = (f_{|p})_*\alpha \in M(w)$ otherwise. The map is then extended by linearity.

2. If f is of constant relative dimension d a pullback $f^* : C_p(Y) \to C_{p+d}(X)$ is defined this way: for a cycle α with support on a point p put $Z = X \times_Y p$. We define $f^*(\alpha)$ as $\sum_{w \in Z_d} l(w) f^*_{|w}\alpha \in C_d(Z)$, where $l(w)$ is the length of $\mathscr{O}_{Z,w}$. The map is then extended by linearity.

   If we can use the codimension notation, we get $f^* : C^p(Y) \to C^p(X)$.

While we defined pullback and pushforward in general, these will not of course in general commute with the differential or compose in a reasonable way. The next proposition shows that this happens in the cases we're interested in, that is, flat pullback and proper pushforward.

We would like to underline that we are not asking any representability condition on our maps, as defined in general in [Sta15, 05VT,03ZL].

Proposition 2.2.

- *If f is flat, then $f^* \circ \mathrm{d}_Y = \mathrm{d}_X \circ f^*$.*
- *If f is proper, then $f_* \circ \mathrm{d}_X = \mathrm{d}_Y \circ f_*$.*

- *If* $f : X \to Y, g : Y \to Z$ *are flat, then* $f^* \circ g^* = (g \circ f)^*$
- *If* $f : X \to Y, g : Y \to Z$ *are proper, then* $g_* \circ f_* = (g \circ f)_*$
- *Suppose we have a cartesian diagram:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

Where f, f' *are flat maps and* g, g' *are proper. Then* $g_* \circ f^* = f'^* \circ g'_*$.

Proof. 1. Let y be a point of Y. We can restrict to the case $y \in Y^{(1)}$. As the statement is local, we can suppose that the fiber of y is irreducible, with generic point $x \in f^{-1}(y)^{(0)}$. Consider now a Nisnevich neighbourhood U of y, with U a scheme. The fiber product $U \times_Y X$ is a Nisnevich neighbourhood of x, and the second projection is a representable map. Let now $W \to X$ be a Nisnevich Neighbourhood of x with W a scheme, the fiber product $U' := W \times_X (U \times_Y X)$ is a scheme, the projection to X is a Nisnevich neighbourhood of x, and the map $U' \xrightarrow{f'} U$ is flat. By applying the result for schemes [Ros96, 4.6] we see that $f'^* \circ \mathrm{d}_U = \mathrm{d}_{U'} \circ f'^*$ which implies that $\delta_x^{\xi_X} \circ f^* = f^* \circ \delta_y^{\xi_Y}$. This proves our statement.

2. Observe that the normalization is a proper map and it satisfies the pushforward rule by the very definition of our differentials, so we can take X, Y to be normal. Let x be a point of $X_{(d)}$ and $y = f(x)$. There are two cases that need to be examined: $y \in Y_{(d-1)}$ and $y \in Y_d$. In both cases we can restrict to $X = \overline{\{x\}}, Y = \overline{\{y\}}$.

   In the first case the formula states that map $f_* \circ \mathrm{d}_X$ must be zero when restricted to $C_d(X) = M(x)$. Consider a point $x' \in X_{(d-1)}$: if $f(x')$ is not y then the map cannot be not finite and we can ignore it. So we can take $Y = y, X = f^{-1}(y)$, but then X is one-dimensional and proper, so it must be a scheme [Sta15, 0ADD] and we can use the result for schemes.

   Suppose now that $y = f(x)$ belongs to $Y_{(d)}$. Consider a point $y' \in Y_{d-1}$. We want to show that $f_* \sum_{f(x')=y'} \delta_{x'}^{x} = \delta_{y'}^{y} \circ f_*$. Let U be a Nisnevich neighbourhood of y', and form the fiber product $Z = X \times_U Y$. Then Z is a Nisnevich neighbourhood of any point in the fiber of y', and the map $Z \to U$ is finite. But an algebraic space that is finite over a scheme must be a scheme (a consequence of the Stein factorization [Sta15, 03XX]), so we can use the result for schemes to conclude.

3. Let $z \in Z$ be a point. To verify the formula we can reduce to $Z = \{z\}$, X and Y the fibers of z. Additionally, as the formula only pertains the value of the maps at the generic points of X and Y we only need to verify it on any Zariski open subset, so we can choose X and Y to be schemes and the result follows.

4. This statement only depends on the properties of Cycle modules, as it can be verified only by looking at maps of spectrum of rings.

5. Let $y \in Y$ be a point, and let $y' \in Y'$ be its image. Again we can reduce to $Y = \{y\}$ and $Y' = \{y'\}$. Then by the same reasoning as above we can substitute X, X' with Zariski open subsets that are schemes, and the result follows.

□

There are two more maps to introduce. The first map is an obvious, but rather useful, multiplication by $\mathscr{O}_X^*$. The second map is a "boundary map" for a closed immersion $Y \to X$.

Definition 2.3. Let X be an algebraic space. We define the following two additional maps:

1. For an element $a \in \mathscr{O}_X^*$, we define the multiplication $\{a\} : C_\bullet(X) \to C_\bullet(X)$ by $a \cdot \sum_{p \in X} \alpha_p = \sum_{p \in X} a_p \cdot \alpha_p$, where a_p is the residue of a at p, seen as an element of degree 1 in Milnor's K-theory ring of $k(p)$. This is a degree 1 map.

2. For a closed immersion $Y \xrightarrow{i} X$, with complement $X \smallsetminus Y \xrightarrow{j} X$ we define the boundary map $\partial_Y^{X \smallsetminus Y} : C_\bullet(X \smallsetminus Y) \to C_\bullet(Y)$ as $i^* \circ \mathrm{d}_X \circ j_*$. The degree of this map is -1, and its total degree is equal to $2(\dim(X) - \dim(Y)) - 3$.

Again we verify that all reasonable compatibilities hold. Note that when we consider operators of odd degree sign appears, as predicted by the Koszul sign rule:

Proposition 2.4. *Let X, Y be algebraic spaces, $V \xrightarrow{i}$ a closed subset of X, $U := X \smallsetminus V \xrightarrow{j} X$ its complement. Then:*

1. *The differential and the boundary map anti-commute: $\mathrm{d}_V \circ \partial_V^U = -\partial_V^U \circ \mathrm{d}_U$.*

2. *The multiplication by $\mathscr{O}_X^*$ and the differential anti-commute: $\{a\}\,\mathrm{d}_X = -\,\mathrm{d}_X\{a\}$.*

   Let now $Y \xrightarrow{f} X$ be a map, and $f^\sharp$ the induced map on structure sheaves.

3. *If f is flat (resp. proper), the pullback (resp. pushforward) and boundary map commute:* $\partial^{Y_U}_{Y_V} \circ f^* = f^* \circ \partial^U_V$ *and* $(f_{|V})_* \circ \partial^{Y_U}_{Y_V} = \partial^U_V \circ (f_{|U})_*$.

4. *If f is flat (resp. proper), the pullback (resp. pushforward) and multiplication by $\mathscr{O}^*_X$ commute:* $\{f^\sharp a\} \circ f^* = f^* \circ \{a\}$ *and* $f_* \circ \{f^\sharp a\} = \{a\} \circ f_*$.

Proof.

- The first property formally descends from the fact that $\mathrm{d}_X \circ \mathrm{d}_X = 0$. We have $\partial^U_V \circ \mathrm{d}_U = i^* \circ \mathrm{d}_X \circ j_* \circ \mathrm{d}_U = i^* \circ \mathrm{d}_X \circ (\mathrm{d}_X \circ j_* - i_* \circ i^* \circ \mathrm{d}_X \circ j_*)$ by direct verification. The first summand is 0 as it contains two consecutive differentials. We can manipulate the second summand obtaining $-i^* \circ \mathrm{d}_X \circ j_* \circ i_* \circ (i^* \circ \mathrm{d}_X \circ j_*) = -i^* \circ \mathrm{d}_X \circ i_* \circ \partial^U_V$. As clearly $i^* \circ \mathrm{d}_X \circ i_* = \mathrm{d}_V$, we can conclude.

- Let x be a point in X^1. We want to show that $\{a\} \circ \delta_x = \delta_x \circ \{a\}$. Consider a Nisnevich neighbourhood $U \xrightarrow{\pi} X$ of x, with a lifting p. Then by definition $\delta_x \circ \{a\} = \pi_* \circ \delta_p \circ \pi^* \circ \{a\}$. We can conclude using the property for schemes and property (4) of this proposition.

- Note that for flat (resp. proper) maps we have the following obvious equalities $(j_Y)_* \circ f^*_{|Y_U} = f^* \circ j_*, i^*_Y \circ f^* = f^*_{|Y_V} \circ i^*$ (resp. $j_* \circ (f_{|Y_U})_* = f_* \circ (j_Y)_*, i^* \circ f_* = f_{|V} \circ i^*_Y$. Using these equalities and proposition (2.2) we can conclude.

- For the statement about flat maps we can reduce to $X = \mathrm{Spec}(k)$ and Y a scheme in the same way as we did in proposition (2.2) and the result follows. The statement for proper maps only needs to be checked on morphisms of spectra of fields and follows directly from the properties of cycle modules.

□

All of these equalities hold at cycle level. We conclude the section by defining the long exact sequence for a pair $U \xrightarrow{j} X$, $V \xrightarrow{i} X$, where V is a closed subset of X and $U = X \setminus V$.

Proposition 2.5. *Let X be an algebraic space, and U, V be as above. Then the sequence:*

$$\ldots \xrightarrow{i_*} A_j(X) \xrightarrow{j^*} A_j(U) \xrightarrow{\partial^U_V} A_{j-1}(V) \xrightarrow{i_*} \ldots$$

Which reads

$$\ldots \xrightarrow{i_*} A_j(X,m) \xrightarrow{j^*} A_j(U,m) \xrightarrow{\partial^U_V} A_{j-1}(V,m-1) \xrightarrow{i_*} \ldots$$

if we write the degree explicitly, is exact.

Proof. It is obvious by definition that the composition of two consecutive maps is zero. Consider an element $\alpha \in \mathrm{Ker}(j^*)$. There is an element $\beta \in C_j(U)$ such that $\mathrm{d}(\beta) = j^*(\alpha)$. Then $\alpha = (\alpha - \mathrm{d}(\beta)) + \mathrm{d}(\beta)$, and $\alpha - \mathrm{d}(\beta)$ belongs to $C_j(V)$. The same reasoning can be applied with very slight changes to prove exactness at ∂_V^U and i_*. □

It is noteworthy that if we take Milnor's K-theory as our Cycle module this exact sequence extends the ordinary sequence for Chow groups, answering the same natural question as Higher Chow groups [Blo86].

The argument of the connection between the two is partially explored in Suslin and Nesterenko's paper [NS89] and in Totaro's phd thesis [Tot92], where they independently prove that for a field F we have $K_n(F) = CH^n(F, n)$. This shows that the Chow groups with coefficients in Milnor's K-theory of a field are contained in its higher Chow groups. The two are not equal, as this would mean that $CH^i(F, j)$ must be zero for $i \neq j$, and for example $CH^2(F, 3)$ can be nonzero. This is shown in the MathOverflow thread [Mat13].

3 General cycle modules and spectral sequences

In his paper, Rost gives a very general definition of cycle module, concerning the points of a given finite type scheme rather than just field extensions of a base field. In our treatment of the subject we decided to restrict to "constant cycle modules over the spectrum of a field" to maintain the analogy with ordinary chow groups, especially in view of the fact that the later chapters of Rost's paper only treat this case. The notion of a general "relative" cycle module will only be needed in this section.

Definition 3.1. Let Y be an algebraic space, and let $\mathrm{Pt}(Y)$ be the category of maps $\mathrm{Spec}(k) \to Y$. A cycle premodule (resp. module) over Y is a contravariant functor from $\mathrm{Pt}(Y)$ to abelian groups satisfying the same properties as an ordinary cycle premodule (resp. module).

We immediately define the object of our interest:

Definition 3.2. Let $Q \xrightarrow{\rho} B$ be a map of finite type of algebraic spaces. For $q \in \mathbb{N}$ We define a cycle (pre)module $A_q[\rho]$ over Y by putting

$$A_q[\rho](p \to Y) = A_q(Q_p)$$

- The pullback map for an extension $P' \xrightarrow{P}$ is the flat pullback $f^* : A_q(X_p) \to A_q(X_{P'})$. If the f is finite we have a pushforward given by the proper pushforward $f_* : A_q(Q_{P'}) \to A_q(Q_P)$.
- The structure of K_*-module of $A_q[\rho]$ is induced by the same structure on A_q.
- Let R be a DVR, with a discrete valuation v, and generic and closed point P, p. Consider a map $\mathrm{Spec}(R) \to Y$. To obtain a differential map $\delta_v : A_q[\rho](P) \to A_q[\rho](p)$ we consider $Z = Q \times_B \mathrm{Spec}(R)$.

  The fiber U of P is open in Z and the fiber V of p is its complement, thus we have a boundary map

  $$(\partial_V^U)_q : A_q[\rho](P) = A_q(U) \to A_q(V) = A_q[\rho](p)$$

  which we take as our δ_v.

Proposition 3.3. *The functor $A_q[\rho]$ we defined is a cycle premodule.*

Proof. All the properties required from cycle premodules descend from the properties we proved in the previous sections for non-relative cycle modules. □

For an algebraic space $X \to B$ over B, and points x, x' we can thus define the local differentials $\delta^x_{x'}$ in the same way as we did in section 1. It remains to check that this collection of data actually defines a cycle module.

Proposition 3.4. *The functor $A_q[\rho]$ we defined is a cycle module.*

Proof.
- (Finite support for divisors) The finite support requirement can be immediately verified for an algebraic space $X \xrightarrow{\phi} B$ by first switching B for the closure of $\phi(X)$, which obviously does not change the cycle groups on X, and then considering an open subset U of B such that B is a scheme, and an open subset X' of X_U that is a scheme. Then the result is true for X', and by Noetherianity it must be true for X.

- (Closedness) We first verify closedness for schemes over B. To do so, we only need to consider local schemes of dimension 2. Let S be such a scheme and s its closed point. Consider a Nisnevich neighbourhood U of the image of s. Then by the same reasoning as (1.4) we can say that the "base changed" result for the cycle module $A_q\left[\rho_U\right]$ obtained (on schemes) by the map $Q \times_B U \xrightarrow{\rho_U} U$ and the map $S \times_B U \to U$ must imply the result for S. Then we can again use the same reasoning as in (1.4) to say that the cycle module on schemes $A_q\left[\rho\right]$ must extend to a cycle module on algebraic spaces.

$\square$

There is an obvious notion of morphism of cycle modules, defined in ([Ros96, 1.3]). While we were not really interested in these morphisms in the case of a cycle module over k_0, here we want to explicitly state the functoriality properties of the construction $A_q\left[\rho\right]$, which are what really makes it a powerful tool.

Proposition 3.5. *Consider a commutative triangle:*

$$\begin{array}{ccc} X & & \\ {\scriptstyle f}\downarrow & \searrow^{\phi} & \\ O & \xrightarrow{\rho} & B \end{array}$$

The following maps are morphisms of cycle modules over B:

1. *If f is proper, we have $[f_*] : A_q\left[\phi\right] \to A_q\left[\rho\right]$ obtained by taking for a point $p \in B$ the pushforward $(f_p)_* : A_q(X_p) \to A_q(O_p)$.*

2. *If f is flat, of relative dimension n, we have $[f^*] : A_q\left[\rho\right] \to A_{q+n}\left[\rho\right]$ obtained by taking for a point $p \in B$ the pullback $f_p^* : A_q(O_p) \to A_{q+n}(X_p)$.*

3. *For a global unit $a \in \mathscr{O}_O^*$ we have $[\{a\}] : A_q\left[\rho\right] \to A_q\left[\rho\right]$ obtained by taking for a point $p \in B$ the multiplication $\{a_{|p}\} : A_q(O_p) \to A_q(O_p)$.*

4. *For a closed immersion $V \xrightarrow{i} O$ with complement $U = O \setminus V \xrightarrow{j} O$ We have $\left[\partial_V^U\right] : A_q\left[\rho \circ j\right] \to A_{q-1}\left[\rho \circ i\right]$ obtained by taking for a point $p \in B$ the boundary map $\partial_{V_p}^{U_p} : A_q(U_p) \to A_{q-1}(V_p)$.*

Proof. The proof for this statement is only a matter of mechanically applying the compatibilities stated in the previous section. $\square$

For a map $X \xrightarrow{\rho} Y$, we want to construct a spectral sequence relating $A_*(X)$ and $A_*(Y, A_q[f])$, in the style of the Leray-Serre spectral sequence. First we define a filtration on $C_\bullet(X)$:

Definition 3.6. Let $X \xrightarrow{\rho} Y$ be a morphism. Put $X_{(n,l)} = \{x \in X_{(l)} \mid \dim(\rho(x), Y) \leq n\}$. We define

$$C_{(l,n)}(\rho) = \bigoplus_{x \in X_{(l,n)}} M(x) \subseteq C_l(X)$$

Then $\ldots \subset C_{(n-1,*)} \subset C_{(n,*)} \subset \ldots \subset C_\bullet(X)$ is a finite filtration of $C_\bullet(X)$. The p-th subquotient of this filtration is $\bigoplus_{u \in Y_{(p)}} C_\bullet(X_u)$.

A filtered differential object has associated a standard spectral sequence (see [HS97, ch.VIII,sec.2]). If we take $C_\bullet(X)$ with the filtration above, the first page is

$$E^1_{p,q} = \bigoplus_{u \in Y_{(p)}} C_q(X_u)$$

The differential $\mathrm{d}^1_{p,q}$ is the map

$$\Theta : \bigoplus_{u \in Y_{(p)}} C_q(X_u) \to \bigoplus_{u \in Y_{(p-1)}} C_q(X_u)$$

given by

$$\Theta^y_{y'} = \sum_{\substack{x \in \rho^{-1}(y) \\ x' \in \rho^{-1}(y')}} \delta^x_{x'}.$$

Proposition 3.7. *There is a convergent spectral sequence:*

$$E^2_{p,q} = A_p(Y, A_q[\rho]) \Rightarrow A_{q+p}(X, M)$$

Proof. It suffices to prove that the differential for the first page of the spectral sequence above is just the differential d_Y for the cycle module $A_q[\rho]$.

First for all x we may remove for the formula for $(\Theta)^y_{y'}$ all x' not belonging to $\overline{\{x\}}^{(1)}$, as the differential $\delta^x_{x'}$ is trivial. By the dimension inequality ([Mat80, p.85]) we may further assume that $y' \in \overline{\{y\}}^{(1)}$. Then let $Z = \mathrm{Spec}(\mathscr{O}_{\overline{\{y\}},y'}) \to Y$, where the local ring is taken in the Nisnevich topology. Then

$$\delta^x_{x'} = (\mathrm{d}_X)^x_{x'} = (\mathrm{d}_{X \times_Y Z})^x_{x'} = (\partial^{X_y}_{X_{y'}})^x_{x'}$$

and the last term is just the component $(\mathrm{d}^y_{y'})^x_{x'}$ of the differential for the cycle module $A_q[\rho]$, so by summing over all points we see that $\Theta = \mathrm{d}_Y$.

□

Let now $X \xrightarrow{\nu} Y$, $X' \xrightarrow{\rho} Y'$ be morphisms. A map of abelian groups $\alpha : C_\bullet(X) \to C_\bullet(X')$ is called filtration-preserving of degree r, t if $\alpha(C_{p,l}(\nu)) \subseteq C_{p+r,l+t}(\rho)$. If such a map is also compatible with the differential, it induces a map on spectral sequences $\alpha : E^2_{p,q}(\nu) \to E^2_{p+r,q+t}(\rho)$. The usual maps are all filtration preserving of some appropriate degree, as we will see in the following lemma:

Lemma 3.8. *Consider a commutative diagram:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \rho} \\ Y & \xrightarrow{f'} & Y' \end{array}$$

1. *We have $f_*(C_{p,l}(\nu)) \subseteq C_{p,l}(\rho)$*

2. *Suppose f has constant relative dimension s and f' has constant relative dimension t. Then $f^*(C_{p,l}) \subset C_{p+s,l+t}(\nu)$.*

3. *if $a \in \mathscr{O}^*(X')$ then $\{a\}(C_{p,l}(\nu)) \subset (C_{p,l}(\rho))$.*

4. *For a couple $V \xrightarrow{i} Q$, $U = Q \setminus V \xrightarrow{j} X$ with i a closed immersion we have $\partial^U_V(C_{p,l}(\rho \circ j) \subseteq C_{p,l-1}(\rho \circ i)$.*

5. *For a couple $V \xrightarrow{i} Y, U \xrightarrow{j} Y$ as above let ν_i be the projection $X_V \to V$ and ν_j the projection $X_U \to U$. We have $\partial^{X_U}_{X_V}(C_{p,l}(\nu_j)) \subset C_{p-1,l}(\nu_i)$.*

Proof. All the statements of the lemma checks are obvious. □

We will use the diagram below to summarize the needed compatibilities in a single proposition.

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & Q \times_B X' & \xrightarrow{f_2} & Q \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \rho} \\ X' & \xrightarrow{\mathrm{Id}} & X' & \xrightarrow{f_3} & B \end{array}$$

Let $f = f_2 \circ f_1$.

Proposition 3.9. *1. If f_1, f_2 are proper the map $f_* : E^2_{p,q}(\nu) \to E^2_{p,q}(\rho)$ equals the composite*

$$A_p(X', A_q[\nu]) \xrightarrow{[f_1]_*} A_p(X', A_q[\rho]) \xrightarrow{(f_2)_*} A_p(B, A_q[\rho])$$

2. Suppose the diagram above is flat and put $r = \dim(f_2), s = \dim(f_1)$. Then the map $f^* : E^2_{p,q}(\rho) \to E^2_{p+r,q+s}(\nu)$ is equals the composite:

$$A_p(B, A_q[\rho]) \xrightarrow{f_3^*} A_{p+r}(X', A_q[\rho]) \xrightarrow{[f_2]^*} A_{p+r}(X', A_{q+s}[\nu])$$

3. For $a \in \mathscr{O}^*(O)$ the map $\{a\} : E^2_{p,q}(\rho) \to E^2_{p,q}(\rho)$ corresponds to $[\{a\}]$.

4. for a couple $V \xrightarrow{i} Q$, $U = Q \setminus V \xrightarrow{j} X$ with i a closed immersion the map $\partial^U_V : E^2_{p,q}(\rho \circ j) \to E^2_{p,q-1}(\rho \circ i)$ corresponds to $\left[\partial^U_V\right]$.

5. for a couple $V \xrightarrow{i} B, U \xrightarrow{j} B$ as above let ν_i be the projection $Q_V \to V$ and ν_j the projection $Q_U \to U$. Then the map $\partial^{Q_U}_{Q_V} : E^2_{p,q}(\rho_j) \to E^2_{p-1,q}(\rho_i)$ is equal to $\partial^{Q_U}_{Q_V} : A_p(Q_U, A_q[\rho]) \to A_{p-1}(Q_V, A_q[\rho])$

Proof. As for much of this section, the original proof in Rost's paper can be applied without any further comments. □

Recall now that for any cycle module M, and field F there is an exact sequence [Ros96, 2.22,sec.2]

$$0 \to M(\mathrm{Spec}(F)) \xrightarrow{\pi^*} M(\mathrm{Spec}(F(t))) \xrightarrow{\mathrm{d}_{\mathbb{A}^1_F}} \bigoplus_{x \in \mathbb{A}^{1\,(1)}_F} M(x) \to 0$$

Here $\mathbb{A}^1_F = \mathrm{Spec}(F[t])$, and π is the extension $\mathrm{Spec}(F(t)) \to \mathrm{Spec}(F)$. This shows that $A^i(\mathbb{A}^1_F) = 0$ for $i \neq 0$, and by induction we see that the same is true for $\mathbb{A}^n_F$, for any field F. We can now combine this statement with the spectral sequence we described before to obtain the homotopy property:

Proposition 3.10. *Let X, E be algebraic spaces, and let $E \xrightarrow{\pi} X$ be an affine bundle, that is, there exists an étale covering U of X such that $E \times_X U$ is isomorphic as a scheme over U to $\mathbb{A}^n \times U$ and the transition maps are affine.*

Then $\pi^* : A^i(X) \xrightarrow{\simeq} A^{i+n}(E)$ is an isomorphism.

Proof. Note first that an affine bundle is necessarily Nisnevich-locally trivial, by the speciality of the group Aff_n. This shows that the fiber of a point $\mathrm{Spec}(k) \to X$ is always isomorphic to $\mathbb{A}^n_k$. By the remark above the cycle modules $A_q[\pi]$ are zero whenever $q \neq n$. This shows that the spectral sequence $A_p(X, A_q[\pi])$ collapses, as only the nth column is non-zero.

We now apply proposition 3.9 using the following diagram:

$$\begin{array}{ccccc} E & \xrightarrow{\pi} & X & \xrightarrow{\mathrm{Id}} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ X & \xrightarrow{\mathrm{Id}} & X & \xrightarrow{\mathrm{Id}} & X \end{array}$$

The statement is that the map $\pi^* : E^2_{p,q}(\mathrm{Id}) \to E^2_{p,q+n}(\pi)$ is equal to the composite of $\mathrm{Id}^* : A_p(X, A_q[\mathrm{Id}]) \to A_p(X, A_q[\mathrm{Id}])$ and $[\pi]^* :\to A_p(X, A_q[\mathrm{Id}]) \to A_p(X, A_{q+n}[\pi])$.

As both the spectral sequences for π and the identity collapse, the map $\pi^* : E^2_{p,q}(\mathrm{Id}) \to E^2_{p,q+n}(\pi)$ is none other than the usual map $\pi^* : A_p(X) \to A_{p+n}(E)$ and using the statement above proving the proposition reduces to proving that the morphism of cycle modules $[\pi]^*$ is an isomorphism. Then the question concerns points $\mathrm{Spec}(k) \to X$ and the homotopy property gives an immediate positive answer. □

4 Gysin map, pullback, intersection

In keeping faithful to our cycle-driven treatment of the subject, we will define an explicit inverse of the pullback map along an affine bundle, depending on a trivialization of our bundle with some additional information.

Given an n-dimensional affine bundle $\pi : E \to X$, and some additional information σ, we will define a map $r_\sigma : C_p(E) \to C_{p-n}(X)$ such that $r_\sigma \circ \mathrm{d}_E = \mathrm{d}_X \circ r_\sigma$ and $r_\sigma \circ \pi^*$, and a map $H_\sigma : C_p(E) \to C_{p+1}(E)$ of degree 1 such that $H_\sigma \circ \pi^* = 0$ and $\mathrm{d}_E \circ H + H \circ \mathrm{d}_E = \mathrm{Id}_E - \pi^* \circ r_\sigma$. In the language of Homology, we are constructing a homotopy between the identity and $\pi^* \circ r_\sigma$.

Definition 4.1. Let $E \xrightarrow{\pi} X$ be an affine bundle. A coordination σ for π is a sequence $X_0 = X \supset X_1 \supset \ldots \supset X_l = \emptyset$ of closed subsets of X such that $E \times_X (X_i \smallsetminus X_{i+1}) \to X_i \smallsetminus X_{i+1}$ is a trivial bundle.

Coordinations clearly always exist because an algebraic space with quasi compact diagonal is generically a scheme.

We would like to underline that up to refining a given coordination we can equip it with a stronger set of data, that is, we can take a couple $(\sigma, \sqcup U_i \to X)$ where

$\sqcup U_i \to X$ is a Nisnevich cover of X trivialization π and the inclusion of $X_i \setminus X_{i+1}$ in X has a lifting to U_i. The construction of r_σ will be in a way local with respect to the covering U_i, and r_σ can be completely computed on the trivialization by keeping track of the transition maps, which may help to clarify the process.

We begin by describing the maps r_σ, H_σ for a trivial line bundle with the trivial coordination. We will refer to this particular case by calling our map r_{triv}, H_{triv}. We define them as the following compositions:

$$r_{triv} = C_p(X \times \mathbb{A}^1) \xrightarrow{j^*} C_p(X \times (\mathbb{A}^1 \setminus \{0\})) \xrightarrow{-\frac{1}{t}} C_p(X \times (\mathbb{A}^1 \setminus \{0\})) \xrightarrow{j'_*} \\ C_p(X \times (\mathbb{P}^1 \setminus \{\infty\})) \xrightarrow{\partial_\infty} C_{p-1}(X)$$

Here j is the open immersion, we write $\mathbb{A}^1 = \mathrm{Spec}(k_0\,[t])$, the map j' is the inclusion of $X \times (\mathbb{A}^1 \setminus \{0\})$ in $X \times (\mathbb{P}^1 \setminus \{\infty\})$ and ∂_∞ is the boundary map for the inclusion of $X \times \{\infty\}$.

$$H_{triv} = C_p(X \times \mathbb{A}^1) \xrightarrow{\mathrm{Pr}_2^*} C_{p+1}(X \times (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta)) \xrightarrow{s-t} \\ C_{p+1}(X \times (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta)) \xrightarrow{\mathrm{Pr}_{1*}} C_{p+1}(X \times \mathbb{A}^1)$$

Here Δ is the diagonal of $\mathbb{A}^2 = \mathrm{Spec}(k_0\,[s,t])$, and $\mathrm{Pr}_1, \mathrm{Pr}_2$ are the projections $X \times (\mathbb{A}^1 \times \mathbb{A}^1 \setminus \Delta) \to X \times \mathbb{A}^1$.

These two maps satisfy the requirements listed above, and the proof in [Ros96, 9.1] can be used without any change. Now we can define by induction the maps for the general trivial affine bundle, which we will again call r_{triv}, H_{triv} when it leads to no confusion. Suppose we have defined maps $r_{triv,i}, H_{triv,i}$ for $\mathbb{A}^i \times X \to X$, with $1 \leq i < n$. We decompose $X \times \mathbb{A}_n \to X$ as $X \times \mathbb{A}_n \xrightarrow{\rho} X \times \mathbb{A}^1 \to X$, and we define:

$$r_{triv,n} = r_{triv,1} \circ r_{triv,n-1}$$

$$H_{triv,n} = H_{triv,n-1} + \rho^* \circ H_{triv,1} \circ r_{triv,n-1}$$

These again verify the required properties by [Ros96, 9.2].

We consider now an affine bundle $E \xrightarrow{\pi} X$ of dimension n, with a coordination ρ of length l. We will proceed inductively on l. If $l = 1$, we are in the case above of a trivial bundle with a trivial coordination. Suppose that we can construct r_σ for any coordination σ of length at most $l - 1$. We can formally write

$$C_p(X) = C_p(X \setminus X_1) \oplus C_p(X_1)$$

Where X_1 is the first nontrivial closed subset in σ. Write $E_1 = E \times_X X_1$ With the above decomposition in mind we write:

$$r_\sigma = \begin{pmatrix} r_{triv} & 0 \\ r_{\sigma|X_1} \circ \partial_{E_1}^{E \smallsetminus E_1} \circ H_{triv} & r_{\sigma|X_1} \end{pmatrix}$$

and

$$H_\sigma = \begin{pmatrix} H_{triv} & 0 \\ H_{\sigma|X_1} \circ \partial_{E_1}^{E \smallsetminus E_1} \circ H_{triv} & H_{\sigma|X_1} \end{pmatrix}$$

It is again straightforward to verify that these maps fit our needs. Recapping, we construct the map r_σ by starting from the bottom of our coordination, where we can use the formulas for a trivial bundle, and work our way up using the "gluing formulas" above.

With the next proposition we explicitly state some compatibilities that are only hinted in the original paper.

Proposition 4.2. *Consider a cartesian diagram*

$$\begin{array}{ccc} E' & \xrightarrow{F} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

With π an affine bundle. Given a coordination σ for E, let $\sigma' = f^{-1}(\sigma)$ the coordination for E' consisting of the inverse images of the elements of σ. Then:

1. *If f is flat, then $r_{\sigma'} \circ F^* = f^* \circ r_\sigma$.*
2. *If f is proper, then $f_* \circ r_{\sigma'} = r_\sigma \circ F_*$.*
3. *For a unit $a \in \mathscr{O}_X^*(X)$, we have $\{a\} \circ r_\sigma = r_\sigma \circ \{\pi^\sharp(a)\}$.*

The equalities above hold at cycle level. Moreover, for a closed immersion $V \to X$, with complement $U \to X$, we have $\partial_V^U \circ r_\sigma = r_\sigma \circ \partial_{E_V}^{E_U}$. This last equality holds at homology level.

Proof. We only need to check that $f^*, f_*, \{a\}$ commute with each of the pieces of the maps we defined. It suffices to apply propositions 2.2 and 2.4 to see that each of these maps commutes in the appropriate sense with both r_{triv}, H_{triv} and the gluing maps defined above.

To prove the last statement, it suffices to see that at homology level $r_\sigma = (\pi^*)^{-1}$ and use the formula $\partial^{E_U}_{E_V} \circ \pi^* = \pi^* \circ \partial^U_V$. $\square$

Finally, we add a last proposition to verify that r_σ and H_σ are filtration preserving map:

Proposition 4.3.

- *Let $E \xrightarrow{\pi} X$ be an affine bundle of dimension n, with a coordination σ, and let $f : X \to Y$ be a map. Then*

$$r_\sigma(C_{p,l}(f \circ \pi)) \subset C_{p,l-n}(f)$$

$$H_\sigma(C_{p,l}(f \circ \pi)) \subset C_{p,l+1}(f \circ \pi)$$

- *Let*

$$\begin{array}{ccc} E' & \xrightarrow{F} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow{f} & X \end{array}$$

be a cartesian diagram, with E as above, and let $\sigma' = f^{-1}(\sigma)$. Then

$$r_{\sigma'}(C_{p,l}(F)) \subset C_{p-n,l-n}(f)$$

$$H_{\sigma'}(C_{p,l}(F)) \subset C_{p+1,l+1}(F)$$

Proof. This can be verified by following the construction for r_σ, H_σ. $\square$

We will use this retraction to construct a pullback for any map to a smooth algebraic space. To do this, we first need to introduce the normal cone, deformation space and double deformation space for a closed imbedding $Y \to X$. Following the original paper we will give the constructions for affine schemes, and the construction for algebraic spaces will follow by standard étale descent arguments.

Definition 4.4. Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let $\mathrm{Spec}(A/I) = Y \xrightarrow{i} X$ be a closed imbedding.

- The normal cone to Y in X, denoted $N_Y X$ is the spectrum of the ring $O_N = \bigoplus_{n \geq 0} {}^{I^n}\!/\!_{I^{n+1}}$.

  It is an ${}^{A}\!/\!_{I}$-algebra and the projection to the zero degree summand gives an homomorphism $O_N \to {}^{A}\!/\!_{I}$.

- The deformation space $D(X, Y)$ is defined as the spectrum of the subring $O_D = \langle I^n t^{-n} \rangle_{n \geq 0}$ of $A[t, t^{-1}]$.

  The subring O_D is finitely generated over $A[t]$, and $O_D[t^{-1}] = A[t, t^{-1}]$, implying that O_D is flat over $k_0[t]$. The quotient ${}^{O_D}\!/\!_{tO_D} = \bigoplus_{n \geq 0} {}^{I^n}\!/\!_{I^{n+1}}$ is isomorphic to O_N.

- Consider a composition of closed embeddings $\mathrm{Spec}({}^{A}\!/\!_{J}) = Z \xrightarrow{i'} Y \to X$. The double deformation space $\overline{D}(X, Y, Z)$ is defined as the spectrum of the subring $O_{\overline{D}} = \langle I^n J^m t^{-n} s^{-m} \rangle_{n,m \geq 0}$ of $A[t, s, t^{-1}, s^{-1}]$.

We will not describe the properties of the double deformation space as it is only used as tool to check the associativity of some operations, and we are only interested in explicitly stating that the construction behaves well for algebraic spaces.

We can restate the properties described in the definition by saying that the normal cone $N_Y X$ has projection to Y and an embedding $Y \xrightarrow{\sigma} N_Y X$. Moreover, if i is a regular imbedding of codimension d the normal cone is a vector of dimension d bundle over Y with zero section the embedding σ.

Similarly, the deformation space $D(X, Y)$ is a scheme over $X \times \mathbb{A}^1$, flat over $\mathbb{A}^1$. The fiber of $\mathbb{A}^1 \setminus \{0\}$ is equal to $X \times (\mathbb{A}^1 \setminus \{0\})$, and the fiber of $\{0\}$ is equal to $N_Y X$.

Proposition 4.5. *The three constructions defined above extend to local immersions of algebraic spaces, as well as all the properties described. Moreover, the properties* $(10.0.1), \ldots, (10.0.5)$ *in [Ros96, sec.10] hold for the double deformation space.*

Proof. Given an étale covering $U \to X$ by an affine scheme all the constructions above define étale descent data in an obvious way as they are defined naturally from the sheaves of ideals I_Y and I_Z associated to the closed imbeddings $Y \to X, Z \to X$. These data are always effective as algebraic spaces form a *fppf* stack [LMB99, 10.7]. Lastly, all the properties we stated for our constructions are base-and-target étale local [Sta15, 02YJ, 036J, 036M]. □

For a locally closed immersion $Y \xrightarrow{i} X$ we define a map $j(i) : C_p(X) \to C_p(N_Y X)$:

Definition 4.6. Let $Y \xrightarrow{i} X$ be a locally closed immersion. We see $X \times (\mathbb{A}^1 \setminus \{0\})$ as the inverse image of $\mathbb{A}^1 \setminus \{0\}$ in $D(X,Y)$, and we name π the projection $D(X,Y) \to X$. We define $j(i) : C_p(X) \to C_p(N_Y X)$ as the composition:

$$j(i) = C_p(X) \xrightarrow{\pi^*} C_{p+1}(X \times (\mathbb{A}^1 \setminus \{0\})) \xrightarrow{\{t\}} C_{p+1}(X \times (\mathbb{A}^1 \setminus \{0\})) \xrightarrow{\partial} C_p(N_Y X)$$

Where ∂ is the boundary map for the couple

$$N_Y X \to D(X,Y), \quad X \times (\mathbb{A}^1 \setminus \{0\}) \to D(X,Y).$$

We will skip all the technical lemmata present in [Ros96, Sec.11-13], and consequently omit most of the proofs. The arguments are highly technical but translate directly and completely to the setting of algebraic spaces. the next proposition shows that the construction of J is compatible with the basic maps

Proposition 4.7. *Consider a locally regular immersion of codimension d $V \xrightarrow{i} X$ and let $f : Z \to X$ be a morphism. Form a cartesian square:*

$$\begin{array}{ccc} Z_V & \xrightarrow{f_1} & V \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i} \\ Z & \xrightarrow{f} & X \end{array}$$

Let $N(f) : N_{Z_V} Z = N_V X \times_X Z \to N_V X$ be the first projection. Then:

- *If f is flat and i_1 is locally a regular immersion of codimension d, then $N(f)^* \circ J(i) = J(i_1) \circ f^*$.*
- *If f is proper and i_1 is locally a regular immersion of codimension d, then $N(f)_* \circ J(i_1) = J(i) \circ f_*$.*

Proof. We can check the compatibilities using the functoriality of the deformation space and propositions (2.2,2.4). □

In what follows all algebraic spaces are equidimensional, and we will switch to the codimension notation, which is more natural in this setting.

Definition 4.8. Let X be an algebraic space smooth over k_0. Then the tangent space TX is a vector bundle over X. For a morphism $Y \xrightarrow{f} X$ consider the factorization:

$$f = Y \xrightarrow{i} Y \times X \xrightarrow{\text{Pr}_2} X$$

With $i = (\mathrm{Id}, f)$. Then i is locally a regular imbedding and the normal cone $N_Y(X \times Y)$ is equal to the pullback $f^*(TX)$. We choose a coordination σ for TX and define:

$$I_\sigma(f) = r_{f^{-1}\sigma} \circ J(i) \circ \mathrm{Pr}_2^* : C^p(X) \to C^p(Y)$$

The map $I_\sigma(f)$ commutes with the differential, and we will denote the induced map on $A^\bullet$ by f^*.

We are creating a possible notational problem here, as if f is flat there is also another map we call f^*. The following proposition will show that this is not the case.

Proposition 4.9. *Let X, Y be smooth over k_0, and consider maps $Z \xrightarrow{f} Y \xrightarrow{g} X$.*

1. *We have $(g \circ f)^* = f^* \circ g^*$.*
2. *If f is flat then $I_\sigma(f)$ and the flat pullback f^* induce the same maps in homology.*
3. *If $f : Y \to X$ is dominant then $f^*_{|C^0(X)} = f^*_{|\xi_X} : M(\xi_X) \to M(\xi_Y)$.*
4. *If $Y \xrightarrow{i} X$ is a regular imbedding, then i^* is homotopic to $r_\tau \circ J(i)$, where τ is any coordination of the vector bundle $N_Y X \to Y$.*
5. *Consider a cartesian diagram:*

$$\begin{array}{ccc} Y & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} \\ X & \xrightarrow{f} & X' \end{array}$$

With h smooth and proper and X' smooth over k_0. Then $h_* \circ f'^* = f^* \circ h'_*$.

Proof. See [Ros96, 12.1-12.5]. Point (3) is a corollary of (2) after restricting to an open subset. □

We can describe very explicitly the pullback to a point for an algebraic space X smooth over k_0. Given a DVR R with a parameter t and valuation v we can define a new map:

$$s_v^t : M(\mathrm{Spec}(k(R))) \to M(\mathrm{Spec}(k(v))), \quad s_v^t(\alpha) = \delta_v(t \cdot \alpha)$$

Proposition 4.10. *Let X be an algebraic space smooth over k_0, and $p \in X^{(1)}$. Then the pullback $C^0(X) \to C^0(p) = M(p)$ is equal to s_v^t where v is the valuation induced by the local Nisnevich ring at p and t is any parameter.*

Let X be as above, let $p \in X^{(r)}$ be a point with a Nisnevich neighbourhood $U \xrightarrow{\pi} X$, and $t_1, \ldots, t_r$ be any regular sequence at p for U. Consider the induced sequence of valuation fields $k(v_1), \ldots, k(v_r)$ with parameters $t_1, \ldots, t_r$. Then the pullback $C^0(X) \to C^0(p) = M(p)$ is equal to $s_{v_r}^{t_r} \circ \ldots \circ s_{v_1}^{t_1} \circ \pi^*$.

Proof. See [Ros96, 12.4]. □

We finally define cross products and an intersection pairing. First note that given an algebraic space X over a field k there is a natural action of $M(\mathrm{Spec}(k))$ on $C_p(X)$ given by $\alpha \cdot \mu = \sum_{p \in X_{(p)}} \alpha_p \cdot \mu$, where α_p is the pullback of $\alpha \in M(\mathrm{Spec}(k))$ through the map $p \to \mathrm{Spec}(k)$.

Definition 4.11. Let Y, Z be algebraic spaces. For a point $y \in Y$, let π_y be the projection $Z \times_Y y = Z_y \to Z$ and let $i_y : Z_y \to Y \times Z$ be the inclusion. Given an element $\rho \in C_p(Y)$ we write ρ_y for the y-component of ρ. We define a cross product:

$$\times : C_p(Y) \times C_q(Z) \to C_{p+q}(Y \times Z), \quad \rho \times \mu = \sum_{y \in Y_{(p)}} (i_y)_*(\rho_y \cdot \pi_y^*(\mu))$$

Equivalently we can take the specular definition:

$$\rho \times \mu = \sum_{z \in Z_{(q)}} (i_z)_*(\pi_z^*(\rho) \cdot \mu_z)$$

Proposition 4.12. *The cross product has the following properties:*

1. *Associativity: $(\rho \times \mu) \times \eta = \rho \times (\mu \times \eta)$ in $C_{p+q+r}(X \times Y \times Z)$.*
2. *Graded-commutativity: if ρ is of pure degree m and μ is of pure degree n then $\rho \times \mu = (-1)^{mn} \mu \times \rho$.*
3. *Chain rule: for ρ, μ as above $\mathrm{d}_{Y \times Z}(\rho \times \mu) = \mathrm{d}_Y(\rho) \times \mu + (-1)^n \rho \times \mathrm{d}_Z(\mu)$.*
4. *Compatibility with pullback, pushforward: for a map $X \xrightarrow{f} Y$ we have respectively $(f \times \mathrm{Id_Z})^*(\rho \times \mu) = f^*(\rho) \times \mu$ if f is flat and $(f \times \mathrm{Id_Z})_*(\eta \times \mu) = f_*(\eta) \times \mu$ if f is proper.*

5. *Compatibility with multiplication by $\mathscr{O}^*$ and boundary maps: for an element $a \in \mathscr{O}_Y^*(Y)$ we have $\{\mathrm{Pr}_1^\sharp a\}\rho \times \mu = (\{a\}\rho) \times \mu$. For an open immersion $U \to Y$ with complement $V \to Y$ we have $\partial_{V\times Z}^{U\times Z}(\rho \times \mu) = (\partial_V^U \rho) \times \mu$ in $A^\bullet(V \times Z)$.*

6. *Compatibility with pullback for a smooth target: let Y be an algebraic space smooth over k_0, and $X \xrightarrow{f} Y$ a morphism. Given unramified cycles $\mu \in A^\bullet(Y), \rho \in A^\bullet(Z)$ we have $(f \times \mathrm{Id}_Z)^*(\mu \times \alpha) = f^*(\mu) \times \rho$ in $A^\bullet(Y \times Z)$.*

Proof. Everything is proven in [Ros96, 14.2-14.5]. We will spend a few words on the compatibilities (4), (5), (6), which are only hinted in the original paper.

Compatibility with pushforward and pullback can be verified pointwise on cycle level, and the same holds for multiplication by $\mathscr{O}^*$. Compatibility with boundary maps is only true at homotopy level: to check it we can write $\partial_{V\times Z}^{U\times Z} = i^* \circ \mathrm{d}_{Y\times Z} \circ j_*$, with j and i the usual immersions. The maps j_* and i^* are both compatible with the cross product, this can be checked pointwise. So we have $\partial_{V\times Z}^{U\times Z}(\mu \times \rho = i^* \circ \mathrm{d}_{Y\times Z} \circ j_*(\mu \times \rho) = i^* \circ \mathrm{d}_{Y\times Z}(\jmath_*(\mu) \times \rho)$. By the chain rule we get $\mathrm{d}_{Y\times Z}(\jmath_*(\mu) \times \rho) = \mathrm{d}_Y(\jmath_*(\mu)) \times \rho \pm j_*(\mu) \times \mathrm{d}_Z(\rho) = \mathrm{d}_Y(\jmath_*(\mu)) \times \rho$ and we can conclude by the compatibility with i^*.

To prove point (6), we begin with the case where f is locally a regular imbedding of codimension d. Note that in this case $D(X \times Z, Y \times Z) = D(X, Y) \times Z$ and the map $Y \times Z \to X \times Z$ factorizes as:

$$A_*(X \times Z) \xrightarrow{\pi^*} A_*(X \times \mathbb{A}^1 \smallsetminus 0) \times Z \xrightarrow{\{-\frac{1}{t}\}} A_*(X \times \mathbb{A}^1 \smallsetminus 0) \xrightarrow{\partial} A_*(N_Y X \times Z = N_{Y\times Z} X \times Z) \xrightarrow{r} A_*(Y \times Z)$$

We have reduced the map to a sequence of manipulations on the first component, allowing us to conclude. The only thing to note is that the compatibility with the retraction r is an immediate consequence of the compatibility with its inverse. Consider now a general map f. Then f^* factorizes as:

$$A_*(X \times Z) \xrightarrow{\mathrm{Pr}_1^*} A_*(X \times Z \times Y \times Z) \xrightarrow{\tilde{f}} A^\bullet(Y \times Z)$$

Where $\tilde{f}$ is the regular imbedding $Y \times Z \to X \times Z \times Y \times Z$.

We can view the map $X \times Z \times Y \times Z \xrightarrow{\mathrm{Pr}_1} X \times Z$ as the product of the first projections $X \times Y \xrightarrow{p} X, Z \times Z \xrightarrow{q} Z$, so that the pullback $\mathrm{Pr}_1^*(\mu \times \rho)$ is equal to $p^*(\mu) \times q^*(\rho)$ by applying twice the compatibility with flat pullback. Then we can factorize the map $\tilde{f}$ as the composition of the diagonal imbedding $Z \to Z \times Z$ times the identity $\mathrm{Id}_{X\times Y}$ and the regular imbedding $Y \times Z \to X \times Y \times Z$. Using the functoriality of the pullback map and the result for a regular imbedding we can then conclude. □

We can now define an intersection pairing for a map $X \xrightarrow{f} Y$, where Y is a smooth algebraic space. In the case $X = Y, f = \mathrm{Id}_Y$ we get the usual ring structure on $A^\bullet(Y)$, and in general the pairing makes $A^\bullet(X)$ into a $A^\bullet(Y)$-module.

Definition 4.13. Let Y be an algebraic space smooth over k_0, and $X \xrightarrow{f} Y$ a morphism. Then the map $(f, \mathrm{Id}_X) : X \to Y \times X$ is locally a regular embedding. We define the intersection pairing $\frown$: $A^\bullet(Y) \times A^\bullet(X) \to A^\bullet(X)$ by the formula:

$$\rho \frown \mu = r_\sigma \circ J(f, \mathrm{Id}_X)(\rho \times \mu)$$

Here σ is the pullback of a coordination of TX to $f^*(TX) = N_X Y$, as in definition (4.9).

Proposition 4.14. *Let X, Y be as in the definition above. Then:*

1. *The pairing $\frown$ for $X = Y$ turns $A^\bullet(X)$ into a graded-commutative ring.*
2. *The ring structure of $A^\bullet(Y)$ turns $A^\bullet(X)$ into a left $A^\bullet(Y)$ graded module.*
3. *If Y, X are smooth over k_0 then f^* is a morphism of rings, and the structure of $A^\bullet(Y)$-algebra of $A^\bullet(X)$ is compatible with the structure of $A^\bullet(Y)$-module.*
4. *If f is proper, and $\alpha \in A^\bullet(Y)$ then $f_*(\alpha \frown \beta) = \alpha \frown f_*(\beta)$, that is, the map f_* is a morphism of $A^\bullet(X)$-modules.*

Proof.

- This is a direct consequence of the corresponding properties for the cross product.
- We can use the two different factorizations for the closed immersion $X \xrightarrow{(\mathrm{Id}_X, f, f)} Y \times X \times X$ and the compatibility with pullback to obtain the equality $\mu \frown (\rho \frown \eta) = (\mu \frown \rho) \frown \eta$ for $\mu \in A^\bullet X, \rho, \eta \in A^\bullet(Y)$.
- We use the compatibility with pullback and the factorization $X \times X \to X \times Y \to Y \times Y$.
- This point is a consequence of the following lemma applied to the square:

$$\begin{array}{ccc} X & \xrightarrow{(f,\mathrm{Id}_X)} & Y \times X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}_Y \times f} \\ Y & \xrightarrow{\Delta} & Y \times Y \end{array}$$

□

Lemma 4.15. *Consider a locally regular embedding $X \to Y$, with Y smooth over k_0, and a proper map $Z \xrightarrow{f} Y$. Form the cartesian diagram:*

$$\begin{array}{ccc} Z_X & \xrightarrow{i_1} & Z \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

Then if i_1 is also a locally regular embedding of the same codimension as i we have $(f_1)_* \circ i_1^* = i^* \circ f_*$.

Proof. This is a consequence of the compatibilities (4.7[2]) and (4.2[2]). □

Having defined the product structure we introduce here the content of section 6 of Rost's original paper.

For a smooth algebraic space the Chow groups with coefficients have a natural interpretation as the Nisnevich cohomology of an appropriate sheaf, and this interpretation is compatible with both the pullback map and the intersection product.

Proposition 4.16. *Let X be an algebraic space smooth over k_0. The Chow groups with coefficients $A^p(X)$ are locally trivial in the Nisnevich topology.*

Proof. This an immediate consequence of theorem [Ros96, 6.1]. □

Definition 4.17. Given a cycle module M, and an equidimensional algebraic space X smooth over k_0 we define the sheaf $\mathscr{M}_X$ on small Nisnevich site of X as the sheafification of the functor $U \xrightarrow{A^0} A^0(U)$.

The definition makes sense as every algebraic space U with an étale map to X must be equidimensional itself.

Lemma 4.18. *The sheaf $\mathscr{M}_X$ is equal to the original functor $U \to A^0(U)$.*

Proof. The pullback to a Nisnevich cover $\sqcup U_i \to X$ is clearly injective as the cover always contains an open immersion $U_{\tilde{i}} \to X$. Given an element $\oplus_i \alpha_i \in A^0(\sqcup U_i)$ which satisfies the gluing conditions we construct an inverse image $\alpha \in A^0(X)$ by taking $\alpha = \alpha_{\tilde{i}}$. To check that the element is unramified it is sufficient to apply the gluing conditions and use the fact that for a point $p \in X^{(1)}$ the element $\oplus_i \alpha_i$ must be unramified at any lifting of p. □

Theorem 4.19. *Let X be an algebraic space smooth over k_0. There are isomorphisms $A^p(X) \to H^p_{Nis}(X, \mathscr{M}_X)$ sending elements of degree d to elements belonging to $H^p_{Nis}(X, \mathscr{M}_X\,[p+d])$. The resulting isomorphism $A^\bullet(X, M) \to H^*_{Nis}(X, \mathscr{M}_X)$ is functorial with respect to morphisms of smooth algebraic spaces and the intersection product on $A^\bullet(X, M)$ is equal to the cup product on $H^*_{Nis}(X, \mathscr{M}_X)$.*

Proof. Let C^i be the functor sending an algebraic space U to $C^i(U)$. These functors are clearly sheaves on the small Nisnevich site of X. Consider the sequence of Nisnevich sheaves on X:

$$0 \to \mathscr{M}_X \xrightarrow{i} C^0 \xrightarrow{\mathrm{d}} C^1 \xrightarrow{d} C^2 \xrightarrow{\mathrm{d}} \dots$$

The sequence is exact at $\mathscr{M}_X$ by definition and everywhere else by 4.16. Moreover, by the Leray-Cartan spectral sequence [Nis89, 1.22.1] the sheaves C^p are acyclic, so we can use the resolution above to compute the cohomology of $\mathscr{M}_X$, leading to the result.

The compatibility with pullback and product is a direct consequence of the locality of these constructions. □

5 Chern classes and equivariant theory

5.1 Chern classes

The original paper notably lacks the definition of a theory of Chern classes "with coefficients". This has been done when M is equal to Milnor's K-theory in chapter 9 of Elman, Karpenko and Merkujev's book [EKM08]. We will extend their idea to all cycle modules and to algebraic spaces. Our approach is slightly more cycle-based than the approach in [EKM08].

Definition 5.1. Let $L \xrightarrow{\pi} X$ be a line bundle. Let σ be a coordination for L and let i be the zero-section imbedding. We define the first Chern class $c_{1,\sigma}(L) : C_p(X) \to C_{p-1}(X)$ of the couple (L, σ) as

$$c_{1,\sigma}(L)(\alpha) = r_\sigma \circ i_*(\alpha)$$

Clearly the choice of a coordination is irrelevant in homology and we will just refer to $c_1(L)$ when we are interested in the induced map in homology. The additional data of the coordinations allows for slightly more precise statements on cycle level when we pull back the coordination together with the line bundle, as we will see:

Proposition 5.2. *Consider a morphism $Y \xrightarrow{f} X$ and form the cartesian square:*

$$\begin{array}{ccc} E & \xrightarrow{f_1} & L \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X \end{array}$$

Let σ' be the induced coordination on Y. Then:

1. *If f is proper then $f_*(c_{1,\sigma'}(L)(\alpha)) = c_{1,\sigma}(L)(f_*(\alpha))$.*
2. *If f is flat then $c_{1,\sigma'}(L)(f^*(\beta)) = f^*(c_{1,\sigma}(L)(\beta))$.*
3. *If $a \in \mathscr{O}^*_X(X)$ then $c_{1,\sigma}(\{a\}(\alpha)) = \{a\}(c_{1,\sigma}(\alpha))$.*

   The following properties are true at homotopy level:

4. *If (U, V) is a boundary couple then $\partial^U_V(c_1(L_{|U})(\alpha)) = c_1(L_{|V})(\partial^U_V(\alpha))$.*
5. *If L, E are two line bundles over X then $c_1(L)(c_1(E)(\alpha)) = c_1(E)(c_1(L)(\alpha))$.*
6. *The first Chern class of $L \otimes E$ satisfies $c_1(L \otimes E)(\alpha) = c_1(L)(\alpha) + c_1(E)(\alpha)$.*

Proof. Properties $(1), (2), (3)$ can be immediately obtained by the compatibilities (4.2, 2.2, 2.4). Property (4) is obtained by (2.4) by writing $r_\sigma = \pi^{-1}$ as homology maps. The last two will need a little more work.

Consider a cartesian square:

$$\begin{array}{ccc} L \times_X E & \xrightarrow{\pi'_1} & E \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ L & \xrightarrow{\pi'} & X \end{array}$$

And name i, i' the zero sections respectively of E and L, and $i_1, {i'}_1$ the zero sections of respectively π and π'.

Then $c_1(L)(c_1(E)(\alpha)) = (\pi'^*)^{-1} \circ {i'}_* \circ (\pi^*)^{-1} \circ i_*$. By the compatibility with proper pushforward we have $(\pi'^*)^{-1} \circ {i'}_* \circ (\pi^*)^{-1} \circ i_* = (\pi'^*)^{-1} \circ ({\pi'}^*_1)^{-1} \circ i_* \circ (i'_1)_*$.

By the functoriality of pullback and pushforward we get the equality $(\pi'^*)^{-1} \circ ({\pi'}^*_1)^{-1} \circ i_* \circ (i'_1)_* = ((\pi' \circ \pi_1)^*)^{-1} \circ (i'_1 \circ i)_*$ which is equal to $((\pi'_1 \circ \pi)^*)^{-1} \circ (i_1 \circ i')_*$

as the two maps are the same, and doing the reasoning above backwards we obtain the desired equality.

For the last equality, recall that there is a flat product map $L \times_X E \xrightarrow{\rho} L \otimes E$ such that the composition of ρ and the projection $\pi'' : L \otimes E \to X$ is the projection $\pi_2 : L \times_X E \to X$.

It is easy to see that if i'' is the zero section of π'' then $\rho^* \circ i''_*(\alpha) = \pi_1^* \circ i'_*(\alpha) + \pi'^*_1 \circ i_*(\alpha)$. As the projections from $E \times_X L$ and $E \otimes L$ to X both induce an isomorphism we know that ρ^* must be an isomorphism too. But then

$$c_1(E \otimes L)(\alpha) = (\pi''^*)^{-1} \circ i''_*(\alpha) = (\pi''^*)^{-1} \circ ((\rho^*)^{-1} \circ \rho^*) \circ i'' = (\pi_2^*)^{-1} \circ \rho^* \circ i''$$

Which is in turn equal to

$$(\pi_2^*)^{-1}(\pi_1^* \circ i'_*(\alpha) + \pi'^*_1 \circ i_*(\alpha)) = c_1(L)(\alpha) + c_1(E)(\alpha)$$

. □

As the maps we defined commute on homology level, we will treat the composition $c_1(L_1) \circ c_1(L_2)$ as a commutative product $c_1(L_1) \circ c_1(L_2) = c_1(L_1) \cdot c_1(L_2)$.

We will first use these properties to give a complete description of the Chow groups with coefficients of the projectivization of a vector bundle:

Proposition 5.3. *Let $P(E) \xrightarrow{p} X$ be the projectivization of a line bundle E of rank r over X. The following formula holds for all p:*

$$A_p(E) = \bigoplus_{\substack{n-i=p \\ i \le r}} c_1(\mathscr{O}_{P(E)}(1))^i (p^*(A_n(X))) \simeq \bigoplus_{p+r \le n \le p} A_n(X)$$

Proof. The pullback $C_p(X) \xrightarrow{p^*} C_p(P(E))$ is injective as the map $P(E) \setminus D(\mathscr{O}_{P(E)}(1)$ is a vector bundle and factors through p. We begin with a trivial vector bundle $E = \mathbb{A}^r \times X$. Note that the blowup $\tilde{E}$ along the zero section 0_E is equal to $\mathscr{O}_{P(E)}(1)$, with $\tilde{0_E}$ as the zero section $P(E) \to \mathscr{O}_{P(E)}(1)$. This construction is compatible with the maps to X.

We first note that the pushforward along the zero section $X \to E$ is zero. To see this it is sufficient to note that in the one-dimensional case it is equal to $\mathrm{d}_E \circ \{t\} \circ \pi^*$ where $\mathbb{A}^1 = \mathrm{Spec}(k_0[t])$ and then factorise $X \to X \times \mathbb{A}^1 \to E$. Using this, by a trivial exact sequence argument the chow groups with coefficients of $E\smallsetminus 0_E$ satisfy the following:

$$A_p(E\smallsetminus 0_E) = A_{p-r}(X) \oplus A_{p-1}(X)\,[1]$$

Where the [1] means the groups are shifted up by one in degree. Consider now the exact sequence induced by $P(E) \to \tilde{E}$. If we identify $A_p(\tilde{E})$ with $A_{p-1}(P(E))$ by any retraction r_σ we get the exact sequence:

$$\dots A_p(P(E)) \to A_p(E\smallsetminus 0_E) \xrightarrow{\partial} A_p(P(E)) \xrightarrow{c_1(\mathscr{O}(1))} A_{p-1}(P(E)) \to \dots$$

The map $\tilde{E} \to E$ is proper, so by the compatibility (2.4) of the pushforward and boundary map we have $\partial_X^{E\smallsetminus 0_E} = p_* \circ \partial_{P(E)}^{\tilde{E}\smallsetminus 0_{\tilde{E}}}$. This shows in particular that any element belonging to $C_p(E\smallsetminus 0_E)$ that is ramified in E must be ramified in $\tilde{E}$ too. Using this and the compatibility with projections to X we have a complete understanding of the maps having $E\smallsetminus 0_E$ either as source or target, which allows us to conclude by computing the groups starting from the top dimension and going down.

We consider now a general vector bundle $E \to X$. Note that there always is an open subset U of X such that the bundle is trivial over U, and such that its complement V is of strictly lower dimension than X. We will compare the exact sequences for U, V and the exact sequence for $P(E)_U, P(E)_V$.

Suppose by induction that the formula for a projective bundle is true for V. For

$$T = P(E), P(E)_U, P(E)_V$$

name

$$\tilde{A}_p(T) = \bigoplus_{\substack{n-i=p \\ i\leq r}} c_1(\mathscr{O}_{P(E)}(1))^i(p_{|T}^*(A_n(X))).$$

By the compatibilities stated in the previous proposition, we have an exact sequence:

$$\dots \tilde{A}_p(P(E)) \to \tilde{A}_p(P(E)_U) \xrightarrow{\partial} \tilde{A}_{p-1}(P(E)_V) \to \tilde{A}_{p-1}(P(E)) \dots$$

The inclusions $\tilde{A}_p(T) \to A_p(T)$ are clearly compatible with the exact sequence. As the inclusion is an isomorphism for both $T = P(E)_U$ and $T = P(E)_V$ by the five lemma it must be an isomorphism also for $P(E)$. □

Proposition 5.4. *Let E be a vector bundle of rank r over X, with a filtration by subbundles $E = E_r \supset E_{r-1} \supset \ldots \supset E_0 = 0$ such that the quotients $L_i = {}^{E_i}\!/_{E_{i+1}}$ are line bundles. Let s be a section of E and let Z be the subset where s vanishes. Then for any $\alpha \in A_*(X)$ the element $\prod_{i=1}^{r} c_1(L_i)(\alpha)$ is equivalent to a cycle in Z.*

Proof. We proceed by induction on r. If $r = 1$ we can see this explicitly as the section gives us a trivialization of the bundle over $X \setminus Z$, and as the retraction r_{triv} composed with the zero section is zero we see that taking Z as the first closed subset in a coordination σ the result follows.

Now take a general E with a filtration by subbundles as above. The section s induces a section s_r of L_r, with zero locus $Y \xrightarrow{i} X$. Given a cycle α we first apply $c_1(L_r)$ obtaining a cycle $i_*\beta$, with $\beta \in C_\bullet(Y)$. Then we can pull E_{r-1} back to Y. It has a section $\tilde{s}$ whose zero locus is exactly Z, and we can conclude by the projection formula. □

This statement can be made precise on cycle level by choosing appropriate coordinations for the line bundles L_i, if needed.

We proceed to define Segre classes and Chern classes:

Definition 5.5. Let $E \to X$ be a vector bundle of rank $e + 1$, and let $P(E) \xrightarrow{p}$ be its projectivization. The i-th Segre class $s_i(E) : A_*(X) \to A_{*-i}(X)$ is defined as the composition:

$$A_*(X) \xrightarrow{p^*} A_{*+e}(P(E)) \xrightarrow{(c_1(\mathcal{O}_{P(E)}(1))^{e+i}} P_{*-i}(P(E)) \xrightarrow{p_*} A_*(X)$$

Proposition 5.6. *The Segre classes satisfy the following properties:*

1. *$s_i(E) = 0$ for $i < 0$ and $s_0(E) = \mathrm{Id}$.*

2. *Given two vector bundles E, F over X, the maps $s_i(E)$ and $s_j(F)$ commute for all i, j.*

3. *Given a map $Y \xrightarrow{f} X$, if f is proper then $f_*(s_i(f^*E)\alpha) = s_i(E)(f_*(\alpha))$ for all $\alpha \in A_*(Y)$ and if f is flat then $f^*(s_i(E)(\beta)) = s_i(f^*E)(f^*(\beta))$ for all $\beta \in A_*(X)$.*

4. *Given a boundary couple U, V we have $s_i(E_{|V})(\partial_V^U(\alpha)) = \partial_V^U(s_i(E_{|U})(\alpha))$.*

5. *If E is a line bundle then $s_1(E) = c_1(E^{-1}) = -c_1(E)$.*

Proof. Point (3) can be directly proven on cycle level after choosing a coordination for $\mathscr{O}_{P(E)}(1)$, with a little diagram chase involving the compatibilities in propositions (5.2,2.2). The same goes for point (4), except it is on homology level.

Point (2) can be again proven using the compatibilities in (5.2,2.2), exactly as in [Ful84, 3.1].

To prove point (1), we use point (3) to restrict to the case of $A_p(X)$, with $p = \dim(X)$. Then the maps $s_i(E), i < 0$ must be zero as they maps to $A_{p-i}(X) = 0$. To prove the case $i = 0$ by (3) we may restrict to an open subset of X, so that E is trivial, and compute it directly using the standard trivialization for $\mathscr{O}_{P(E)}(1)$.

Point (5) results from point (1) and the fact that for a line bundle $P^*E = \mathscr{O}_{P(E)}(-1)$. □

We will again denote the composition of Segre classes as a commutative multiplication. We are ready to define the General Chern classes:

Definition 5.7. Let E be a vector bundle over X. Consider the formal power series $s_t(E) = \sum_{i=0}^{\infty} s_i(E)t^i$. We define the Chern polynomial

$$c_t(E) \in \mathbb{Z}\left[s_0(E), \ldots, s_i(E), \ldots\right]$$

as the inverse of $s_t(E)$. The i-th Chern class $c_i(E) : A_*(X) \to A_{*-i}(X)$ is the i-th coefficient of $c_t(E)$. Explicitly:

$$C_0(E) = \mathrm{Id}, \quad c_1(E) = -s_1(E), \quad c_2(E) = s_1(E)^2 - s_2(E), \ldots$$

Proposition 5.8. *Let $E \to X$ be a vector bundle of rank r. The Chern classes $c_i(E)$ have the following properties:*

1. *For all $i > 0$ we have $c_i(E) = 0$.*

2. *If F is another vector bundle on X then $c_i(E)$ and $c_j(F)$ commute for all i, j.*

3. *Let $Y \xrightarrow{f} X$ be a morphism. If f is proper then $f_*(c_i(f^*E)\alpha) = c_i(E)(f_*(\alpha))$ for all $\alpha \in A_*(Y)$ and if f is flat then $f^*(c_i(E)(\beta)) = c_i(f^*E)(f^*(\beta))$ for all $\beta \in A_*(X)$.*

4. *Given a boundary couple U, V we have $c_i(E_{|V})(\partial_V^U(\alpha)) = \partial_V^U(c_i(E_{|U})(\alpha))$.*

5. *Given an exact sequence of vector bundle over X:*

$$0 \to E' \to E \to E'' \to 0$$

We have $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$.

6. *Suppose X is normal. If $E = \mathscr{O}(D)$ for an irreducible subvariety D of codimension 1, defined by a valuation v on $k(X)$, then the restriction of $c_1(E)$ to $A^0(X)$ is equal to the map s_v^t as in proposition (4.10). In particular, $c_1(E)(1) = 1_D$.*

7. *The Chern class $\zeta = c_1(\mathscr{O}_{P(E)}(1))$ satisfies the equation*

$$\zeta^r + \zeta^{r-1}c_1(p^*E) + \ldots + c_r(E) = 0$$

Proof. Properties (2), (3), (4) come directly from the corresponding properties of Segre classes.

Properties (1), (5) can be proven using the splitting construction exactly as done in [Ful84, 3.2].

Finally we prove property (6) by explicit computation. Consider a coordination σ for $L = \mathscr{O}(D)$ with $X_1 = D$. Given an element $\alpha \in A^0(X)$ we can explicitly write down $c_{1,\sigma}(L)(\alpha)$. It is equal to $r_{\sigma_{|D}} \circ \partial_{L_D}^{L_{X \smallsetminus D}} \circ H_{triv} \circ (i_0)_*(\alpha)$.

By explicit verification we see that $H_{triv} \circ (i_0)_*(\alpha) = \{t\}(\pi^*(\alpha))$, where t is the parameter for the trivial bundle over $X \smallsetminus D$. The expression makes sense as the cycle $\pi^*(\alpha)$ is not supported in any point where t is zero.

Now we consider the boundary map $\partial_{L_D}^{L_{X \smallsetminus D}}$. As $\mu = \{t\}(\pi^*(\alpha))$ lives in the generic point of L, the only point where the value of $\partial_{L_D}^{L_{X \smallsetminus D}}(\mu)$ can be nonzero is the generic point of L_D. To compute the map $\delta_{\xi_{L_D}}^{\xi_L}$ we first base change μ to a Nisnevich neighbourhood $U_D \xrightarrow{\rho} X$ of ξ_D such that the bundle is trivial.

In base changing μ to U_D we need to keep track of what happens to $\{t\}$, which is no longer the parameter for our trivial bundle: if t' is the new parameter, we see that $t = \tau t'$, where $\tau \in \mathscr{O}_X^*(U_D \times_X (X \smallsetminus D))$ vanishes in D with order 1. Then the pullback of μ to U_D is equal to $\{t'\}(\rho^*(\pi^*\alpha)) + \{\tau\}(\rho^*(\pi^*\alpha)) = \mu_1 + \mu_2$.

Now it's easy to see that $\partial_{L_D}^{L_{X \smallsetminus D}}(\mu_1) = 0$ and $\partial_{L_D}^{L_{X \smallsetminus D}}(\mu_2) = s_{v_L}^{\tau}(\pi^*(\alpha))$, where v_L is the valuation defining L_D. By the compatibility of the maps s_v^t with the retractions r_σ we obtain the result. □

Corollary 5.9. *If X is smooth over k_0, and D is an irreducible divisor then $c_1(\mathscr{O}(D)) = \frown$*

D and the classes $c_i(E)$ are all equal to $\frown \beta$ for some cycle β of degree zero.

Proof. Let $i : D \to X$ be the inclusion, let $\pi : \mathscr{O}(D) \to X$ be the projection and σ the zero section of π. We need to prove the equation

$$c_1(\mathscr{O}(D)) = \pi^{*-1} \circ \sigma_* = i_* \circ i^*$$

For an element belonging to the image of i_* we do the following: first we compose each side with π^*, obtaining

$$\sigma_* = \pi^* \circ i_* \circ i^* = (i_{\mathscr{O}(D)_D})_* \circ \pi_D^* \circ i^* = (i_{\mathscr{O}(D)_D})_* \circ J(i)$$

We can now apply lemma [Ros96, 11.1], which says that $J(i) \circ i_* = (\sigma_{N_D X})_*$, where $\sigma_{N_D X}$ is the zero section of $N_D X$ to obtain the result.

Let now α be an element not belonging to the image of i_*. By linearity we only have to prove the result for an irreducible α, so we may see $\alpha \in M(p)$ as the pushforward of an element $\alpha \in A^0(\overline{p})$. We can use the projection formula to compare the two sides of the equation on $A^\bullet(\overline{p})$, and point (6) of the previous proposition, together with the compatibilities (4.7) allow us to conclude.

The general case is a direct consequence of the line bundle case by using the splitting principle and the Whitney sum formula. □

Lastly, we add a consideration on the top Chern class of vector bundle. The way we defined the first Chern class of a line bundle can be used to define the top Chern class of a vector bundle in general. The following statement shows that our choice of definitions is not contradictory:

Proposition 5.10. *Let $E \xrightarrow{\pi} X$ be a r-dimensional vector bundle, and let s be the zero section of π. Then $c_r(E) = \pi^{*-1} \circ s_*$.*

Proof. We can just follow step by step the proof in Fulton's book [Ful84, 3.3,3.3.2]. □

As a corollary of the previous result we can describe a class of morphisms $X \to Y$ having a nice property: the Chow groups with coefficients of X can be obtained from those of Y just by looking at the zero degree component, which in a way means that we only have to know what happens for ordinary Chow groups.

Proposition 5.11. *Let $X \xrightarrow{f} Y$ be algebraic spaces smooth over k_0. Consider a morphism $Y \xrightarrow{g} Z$ which induces a pullback map, i.e. g is flat or Z is smooth over*

k_0. Suppose that f can be factorized as a composition of projective bundles and affine bundles. Then the following holds:

$A^\bullet(Y)$ is generated as a ring by the image of $A^\bullet(Z)$ and elements of degree zero if and only if the same is true for $A^\bullet(X)$.

Proof. This is an easy consequence of proposition (5.3) and (3.10). □

We say that $A^\bullet(X)$ is geometrically generated over $A^\bullet(Z)$ if the property above holds. The next corollary shows that this is enough to understand, for example, the Chow groups with coefficients of Grassmanian bundles.

Corollary 5.12. *Let $E \xrightarrow{\pi} X$ be a vector bundle, and let $\mathrm{Gr}_m(E) \to X$ be the grassmann bundle of n-dimensional subbundles of E. Then $A^\bullet(\mathrm{Gr}_m(E))$ is geometrically generated over $A^\bullet(X)$.*

Proof. We can obtain the bundle of flags complete flags $\mathrm{Fl}_m(E)$ from X by a sequence of projective bundles. We begin by considering $P(E)$. Then we consider the vector bundle E_1 on $P(E)$ obtain by quotienting the pullback of E by $\mathscr{O}(-1)$, and take the projectivization $P(E_1)$. This second scheme is clearly isomorphic to $\mathrm{Fl}_2(E)$ as an X-scheme. By repeating this $m-2$ more times we obtain a scheme isomorphic to $\mathrm{Fl}_m(E)$. By the previous proposition, this implies that the Chow groups with coefficients of $\mathrm{Fl}_m(E)$ are geometrically generated over those of X.

Now consider the Grassmann bundle $\mathrm{Gr}_n(E)$ We can take the projectivised $P(V)$ of the tautologic bundle V of $Gr(m,E)$. When we pull back V to a vector bundle V_1 over this space, there is a natural splitting $V_1 = \mathscr{O}(-1) \oplus V_2$. We can do this again for V_1, obtaining another splitting. It is clear that repeating this process yields a space with the same universal property as $\mathrm{Fl}_m(E)$, so that the two must be isomorphic as spaces over X. This implies that the Chow groups with coefficients of $\mathrm{Fl}_m(E)$ are geometrically generated over those of $\mathrm{Gr}_m(E)$ and as the first are geometrically generated over those of X the same holds true for $A^\bullet(\mathrm{Gr}_m(E))$. □

If we use Milnor's K-theory or Galois cohomology as coefficients we can compute the Chow groups with coefficients of $\mathrm{Gr}_m(E)$ just knowing those of X and the ordinary Chow groups of $\mathrm{Gr}_m(E)$, as the zero degree components computed using these Cycle modules are respectively the Chow groups and the Chow groups modulo p.

5.2 Equivariant theory

A cycle-based approach as presented in the previous sections is clearly only reasonable when points have a well defined underlying field. Defining a theory of Chow groups with coefficients for more general algebraic spaces, and most of all for a suitably large class of algebraic stacks will require a different approach. For a quotient stack $[X/G]$ we can use the same type of equivariant approach defined in [Tot99] and [EG96]. This has already been described in [Gui08].

The basic idea is that any extension of our theory should be homotopy invariant, and that the i-th codimensional Chow groups with coefficients should not change if we remove or modify somehow a subset of codimension at least $i+2$. Using this, up to readjusting the codimension index, we can replace our object of study X with another object $E \to X$ that, up to some high codimension subset, is a vector bundle over X.

Definition 5.13. Let i be a positive integer, and let X be an algebraic space with an action by an algebraic group G. Let V be a r-dimensional representation of G such that G acts freely outside of a closed subset $W = V \smallsetminus U$ of codimension equal or greater than $i+2$, and set $U = V \smallsetminus W$.

Consider the quotient $X \times^G U = (X \times U)/G$, where the action of G is the diagonal action. By [Sta15, 02Z2] we know that $X \times^G U$ is an algebraic space, and if X is quasi separated so is $X \times^G U$. In this case we define the i-th codimensional equivariant Chow group with coefficients $A_i^G(X)$ to be $A_{i+r-\dim(G)}(X \times^G U)$.

If X is equidimensional can also switch to the dimensional notation by writing

$$A^i_G(X) = A^G_{r+\dim(X)-\dim(G)-i}(X) = A^i(X \times^G U)$$

This is well defined by the double fibration argument, as in [EG96, 2.2]. Putting it briefly, if we have two representations V, V' of dimension r, r' satisfying the requirements for the definition we can construct a third representation $V \times V'$ and then $A_{i+r+r'}(X \times^G (U \times U'))$ is isomorphic to both $A_{i+r}(X \times^G U)$ and $A_{i+r'}(X \times^G U')$.

Note that there is no reason why the equivariant groups should be zero for codimension $\gg 0$, and in fact this is not the case even for the most basic examples.

Proposition 5.14. *Let X, Y be algebraic spaces equipped with an action of an algebraic group G. Let $f : X \to Y$ be an equivariant morphism. For a representation V of G, with U as above, let $f_U : X \times^G U \to Y \times^G U$ be the induced map. Then if f is* *flat, proper, an open or closed immersion, smooth, a vector bundle of dimension r, or a regular imbedding of codimension d, so is f_U.*

Proof. See [EG96, 2.2, prop.2]. □

Consequently one can see that all of the theory developed in sections $1-5$ immediately translates to the equivariant case. We just have to consider equivariant maps f and consider the induced maps f_U to obtain the desired morphisms on equivariant Chow groups with coefficients.

Corollary 5.15. *Let us fix an algebraic group G. Every result in sections $1-5$ can be restated for G-equivariant maps.*

Proof. Everything follows immediately from the proposition above, and the maps are well defined by the double fibration argument, as in [EG96, 2.2, prop.3]. □

Corollary 5.16. *The equivariant Chow groups with coefficients $A^i_G(X)$ only depend on the quotient stack $[X/G]$.*

Proof. This is proven in [EG96, 5.2, prop.16]. We sketch the proof. Consider two different realizations $[X/G], [Y/H]$ of the same stack, with an isomorphism ϕ. Let V, V' representation respectively of G, H with open subsets U, U' as in definition (5.13). Then $(X \times^G U) \times_{[Y/H]} (Y \times^H U')$ is a scheme, and the second projection is, up to a subset of sufficiently high codimension, a vector bundle. This gives us the desired isomorphism. □

We will now compute some equivariant Chow groups with coefficients, taking G a classical group acting trivially on the spectrum of a field. The computations for $G = GL_n, SL_n$ are an immediate consequence of our description of the Chow ring with coefficients for Grassmanian bundles. In part 2 we will compute the G-equivariant Chow ring with coefficients of a point with $G = \mu_p, O_2, O_3, SO_3$, obtaining some less trivial examples.

Proposition 5.17. *Let our cycle module M be either equal to Milnor's K-theory or Galois cohomology, and let G be the general linear group GL_n or the special linear group SL_n. Then the Equivariant Chow ring with coefficients $A^\bullet_G(\mathrm{Spec}(k))$ is equal to the tensor product of the corresponding ordinary equivariant Chow groups and $M(\mathrm{Spec}(k))$.*

Proof. As in [EG96, 3.1], by choosing an appropriate representation of GL_n we can compute its equivariant Chow groups with coefficients on Grassmann schemes. Then the description given in (5.11) and (5.12) of the Chow groups of Grassamann bundles allows us to conclude immediately. Note that the product structure is the same as that induced on the tensor product by the two products on ordinary Chow groups and on $M(\mathrm{Spec}(k))$; this can be seen as a consequence of proposition (4.10).

Now, following [VM06, sec.3] if we consider the representation of SL_n induced by our original representation of GL_n, we see that the natural map $\mathrm{Spec}(k) \times^{SL_n} U \to \mathrm{Spec}(k) \times^{GL_n} U$ is a G_m-torsor with associated line bundle the determinant bundle. Using the injectivity of c_1, a simple long exact sequence argument shows that the ring $A^{\bullet}_{SL_n}(\mathrm{Spec}(k))$ must be isomorphic to $A^{\bullet}_{GL_n}(\mathrm{Spec}(k))/_{c_1}$ □

Lastly, we extend the result of corollary (4.19) to the equivariant setting.

Proposition 5.18. *Let X be an algebraic space equipped with an action of an algebraic group G. Suppose $[X/G]$ is an algebraic stack smooth over k_0. Then the equivariant Chow ring with coefficients $A^{\bullet}_G(X, M)$ is isomorphic to the cohomology ring $H^{\bullet}_{Sm-Nis}([X/G], A^0(-, M))$.*

The isomorphism sends $H^i_{Sm-Nis}([X/G], \mathrm{Inv}^j(-, M))$ to $A^i_G(X, j-i)$.

Proof. We want to show that given an approximation $X \times^G U$ of $[X/G]$ the pullback through $X \times^G U \to [X/G]$ induces an isomorphism

$$H^n_{Sm-Nis}([X/G], A^0(-, M)) \simeq H^n_{Sm-Nis}(X \times^G U, (A^0(-, M))) \simeq A^n_G(X, M)$$

for $n < d$, where d is as usual the codimension of the complement of U in our chosen representation V minus 1.

Consider the factorization

$$X \times^G U \xrightarrow{i} [(X \times V)/G] \xrightarrow{\pi} [X/G]$$

As the second map is a vector bundle we have $R^i\pi_*((\mathrm{Inv}^{\bullet}(-, M))_{|[(X\times V)/G]}) = 0$ for $i > 0$ and the pullback π^* is an isomorphism due to the Leray spectral sequence.

Using again the Leray spectral sequence we see that the pullback i^* must be an isomorphism for $n < d$: we have

$$R^i i_*((A^0(-, M))_{|X\times^G U}) = (U \to A^i(U, M))^{sm-Nis} = 0 \text{ for } 0 < i < d.$$

□

3. INVARIANTS OF HYPERELLIPTIC CURVES

This chapter is dedicated to computing the cohomological invariants (with coefficients in étale cohomology) for the stacks $\mathscr{H}_g$ of hyperelliptic curves of genus g.

In the first section we set up the main instruments for our calculations, namely the presentation of the stacks $\mathscr{H}_g$ as a quotient by GL_2 if g is even and by PGL_2 if g is odd, and the spaces $\Delta_{i,j}$ of degree j forms divisible by the square of a degree i form.

In the second and third section we compute the cohomological invariants of $\mathscr{H}_g$ for all even genera over an algebraically closed field. We give a different approach for the case of $\mathscr{H}_2$ where we are able to compute the invariants just doing some computations with Chow groups mod p. In the fourth section we partially extend the results for general fields.

The last two sections are dedicated to computing the cohomological invariants of $\mathscr{H}_3$. First we compute a few equivariant Chow groups with coefficients necessary to understand the PGL_2-equivariant Chow groups with coefficients of the projective spaces. We then conclude the computation by techniques similar to those used for the case of g even.

1 Preliminaries

In this chapter we fix a base field k_0 and a prime number p. We will always assume that the characteristic of k_0 is different from p. All schemes and algebraic stacks will be assumed to be of finite type over k_0. If X is a k_0-scheme we will denote by $\mathrm{H}^i(X)$ the étale cohomology ring of X with coefficients in $\mu_p^{\otimes i}$ (here $\mu_p^{\otimes 0} := \mathbb{Z}/p\mathbb{Z}$), and by $\mathrm{H}^\bullet(X)$ the direct sum $\oplus_i \mathrm{H}^i(X)$. If R is a k_0-algebra, we set $\mathrm{H}^\bullet(R) = \mathrm{H}^\bullet(\mathrm{Spec}(R))$.

In this section we state some general considerations that will be needed for all the computations in the chapter.

We begin by recalling the presentations of the stacks we will work with, all due to Vistoli and Arsie [AV04].

Theorem 1.1. *Let g be an even positive integer. Consider the affine space $\mathbb{A}^{2g+3}$, seen as the space of all binary forms $\phi(x) = \phi(x_0, x_1)$ of degree $2g+2$. Denote by X the open subset consisting of nonzero forms with distinct roots. Consider the action of GL_2 on X_g defined by $A(\phi(x)) = \det(A)^g \phi(A^{-1}x)$.*

Denote by $\mathscr{H}_g$ the stack of smooth hyperelliptic curves of genus g. In particular, as any smooth curve of genus 2 is hyperelliptic, $\mathscr{H}_2 = \mathscr{M}_2$. Then we have

$$\mathscr{H}_g \simeq [X_g/GL_2]$$

And the canonical representation of GL_2 yields the Hodge bundle of $\mathscr{H}_g$.

Let g be an odd positive integer. Consider $\mathbb{A}^{2g+3}$ as the space of all binary forms of degree $2g+2$. Denote by X_g the open subset consisting of nonzero forms with distinct roots, and let $PGL_2 \times G_m$ act on it by $([A], \alpha)(f)(x) = \mathrm{Det}(A)^{g+1}\alpha^{-2} f(A^{-1}(x))$.

Then for the stack $\mathscr{H}_g$ of smooth hyperelliptic curves of genus g we have

$$\mathscr{H}_g = [X_g/(PGL_2 \times G_m)]$$

Proof. This is corollary 4.7 of [AV04]. When $g = 2$, the presentation of $\mathscr{M}_2$ was originally shown by Vistoli in [Vis96, 3.1]. □

In both cases, the quotient of X_g by the usual action of $\mathbb{G}_m$ defined by $(x_1, \dots, x_7, t) \to (tx_1, \dots, tx_7)$, which we will name Z_g, is naturally an open subset of the GL_2 (resp. $PGL_2 \times G_m$)-scheme $P(\mathbb{A}^{2g+3})$, namely the complement of the discriminant locus.

Let G be either GL_2 or $PGL_2 \times G_m$. We will first construct the invariants of the quotient stack $[Z/G]$, then use the principal $\mathbb{G}_m$-bundle $[X/G] \to [Z/G]$ to compute the invariants of $\mathscr{H}_g$ for g even and $g = 3$.

Lemma 1.2. *let G be either GL_2 or $PGL_2 \times G_m$. If p differs from 2, the principal $\mathbb{G}_m$-bundle $[X/G] \xrightarrow{f} [Z/G]$ induces an isomorphism on cohomological invariants. If p is equal to 2, it induces an injective map.*

Proof. The statement for $p = 2$ can be immediately deduced by the fact that a $\mathbb{G}_m$-torsor is a smooth-Nisnevich cover.

For $p \neq 2$ we first reduce to the case where G is special, so that $X \to [X/G]$ is a smooth-Nisnevich cover. We already have that GL_2 is special, and for $PGL_2 \times G_m$ we do the following. Consider a representation V of $PGL_2 \times G_m$ satisfying the usual condition that the action is free on an open subset U whose complement has codimension at least 2. Then the quotient $X' = (X \times U/PGL_2)$ is an algebraic space and its cohomological invariants are equal to these of $[X/PGL_2]$. It has two different actions by G_m given by the two actions on X, the first one defined by $(x,t) \to xt^{-2}$ and the second one defined by multiplication. We need to compare the cohomological invariants of $[X'/G_m]$ and those of $[X'/G_m \times G_m]$ when $p \neq 2$.

The cohomological invariants of $[X'/G_m]$ are equal to the invariants of X' satisfying the sheaf condition for the map $X' \to [X'/G_m]$, and the invariants of $[X'/G_m \times G_m]$ are in the same way equal to the invariants of X' satisfying the gluing conditions for $X' \to [X'/G_m \times G_m]$. We want to show that there are no elements in $\mathrm{Inv}^\bullet(X')$ that satisfies the gluing conditions for $X' \to [X'/G_m]$ and does not satisfy the conditions for $X' \to [X'/G_m \times G_m]$.

Denote by $m_1 : X' \times G_m \to X'$ the multiplication relative to the first action, by $m_2 : X' \times G_m \to X'$ the multiplication relative to the second action and by $M : X' \times G_m \times G_m \to X'$ the combined multiplication map. Note that the first multiplication factors through the second as

$$X' \times G_m \xrightarrow{(*^{-2},\mathrm{Id})} X' \times G_m \xrightarrow{m_2} X'$$

Where $*^{-2}$ is the map $G_m \to G:m$ defined by $\lambda \to \lambda^{-2}$.

Consider now a cohomological invariant $\alpha \in \mathrm{Inv}(X')$ satisfying the gluing conditions for m_1, that is, ${\mathrm{Pr}_1}^*(\alpha) = m_1^*(\alpha)$. This happens if and only if we have $((m_1^* - {\mathrm{Pr}_1}^*)(\alpha))(\xi) = 0$, where ξ is the generic point of $X \times G_m$. The extension $\xi \to \xi'$ of generic points induced by $(*^{-2}, \mathrm{Id})$ is a map of degree 2, which implies that it is injective on étale cohomology for $p \neq 2$. As $m_1^* - {\mathrm{Pr}_1}^* = (*^{-2}, \mathrm{Id})^* \circ (m_2^* - {\mathrm{Pr}_1}^*)$ we conclude that $(m_2^* - {\mathrm{Pr}_1}^*)(\alpha)$ must be zero itself, so α also satisfies the gluing conditions for m_2. It is immediate to verify that an element $\alpha \in \mathrm{Inv}(X')$ satisfying the gluing conditions for both m_1 and m_2 must satisfy the gluing conditions for M.

We can repeat the same reasoning almost word by word for the case of GL_2 by using the inclusion $G_m \to GL_2$ to compare the different gluing conditions.

□

We generalize the family of equivariant schemes in Theorem (1.1) this way: let F be the dual of the standard representation of GL_2. We can see F as the space of all binary forms $\phi = \phi(x_0, x_1)$ of degree 1. It has the natural action of GL_2 defined by $A(\phi)(x) = \phi(A^{-1}(x))$. We denote by E_i the i-th symmetric power $\mathrm{Sym}^i(F)$. We can see E_i as the space of all binary forms of degree i, and the action of GL_2 induced by the action on F is again $A(\phi)(x) = \phi(A^{-1}(x))$. If i is even we can consider the additional action of GL_2 given by $A(\phi)(x) = \det(A)^{i/2-1}\phi(A^{-1}(x))$, and the action of PGL_2 given by $[A]\,(\phi)(x) = \mathrm{Det}(A)^{g+1}f(A^{-1}(x))$.

We denote $\tilde{\Delta}_{r,i}$ the closed subspace of E_i composed of forms ϕ such that there exists a form f of degree r whose square divides ϕ. With this notation the scheme X_g in theorem (1.1) is equal to $E_{2g+2} \smallsetminus \tilde{\Delta}_{1,2g+2}$.

We denote $\Delta_{r,i}$ the closed locus of the projectivized $P(E_i)$ composed of forms ϕ such that there exists a form f of degree r whose square divides ϕ. With this notation we have $Z_g = P(E_{2g+2}) \smallsetminus \Delta_{1,2g+2}$.

Thanks to the localization exact sequence on Chow groups with coefficients, understanding the cohomological invariants of $[P^i \smallsetminus \Delta_{1,i}/G]$ can be reduced to understanding the invariants of $[P^i/G]$, which are understood thanks to the projective bundle formula, the top Chow group with coefficients $A^0_G(\Delta_{1,i})$ (which is not equal to the cohomological invariants of $[\Delta_{1,r}/G]$, as $\Delta_{1,i}$ is not smooth) and the pushforward map $A^0_G(\Delta_{1,i}) \to A^1_G(P^i)$. The computation of $A^0_G(\Delta_{1,i})$ will be based on the following two propositions.

Proposition 1.3. *Let $\pi_{r,i} : P(E_{i-2r}) \times P(E_r) \to \Delta_{r,i}$ be the map induced by $(f, g) \to fg^2$. The equivariant morphism $\pi_{r,i}$ restricts to a universal homeomorphism on $\Delta_{r,i} \smallsetminus \Delta_{r+1,i}$. Moreover, if $\mathrm{char}(k_0) > 2r$ or $\mathrm{char}(k) = 0$ then any k-valued point of $\Delta_{r,i} \smallsetminus \Delta_{r,i+1}$ can be lifted to a k-valued point of $P(E_{i-2r}) \times P(E_r)$.*

Proof. See [Vis96, 3.2]. The reasoning holds in general as long as we can say that a polynomial with r double roots must be divisible by the square of a polynomial of degree r. This is clearly true for $\mathrm{char}(k) = 0$, but in positive characteristic it holds only as long as $2r < \mathrm{char}(k)$, as we can find irreducible polynomials of degree $\mathrm{char}(k)$ with only one distinct root. It is however always true that the map $\pi_{r,i}$ is a bijection when restricted to $\Delta_{r,i} \smallsetminus \Delta_{r+1,i}$. Being proper and bijective, it is a universal homeomorphism. □

Proposition 1.4. *The pushforward of a (equivariant) universal homeomorphism induces an isomorphism on (equivariant) Chow groups with coefficients in* $\mathrm{H}^\bullet$*.*

Proof. Note first that the non-equivariant statement implies the equivariant one, as if X, Y are G-schemes on which G acts freely then an equivariant universal homeomorphism between them induces a universal homeomorphism on quotients.

Let $f : X \to Y$ be a universal homeomorphism. Given a point $y \in Y$, its fibre x is a point of X and the map $f_x : x \to y$ is a purely inseparable field extension.

The pullback $(f_x)^* : \mathrm{H}^\bullet(y) \to \mathrm{H}^\bullet(x)$ is an isomorphism, and the projection formula yields $(f_x)_*((f_x)^*\alpha) = [k(x) : k(y)]\,\alpha$. As the characteristic of $k(x)$ is different from p, the degree $[k(x) : k(y)]$ is invertible modulo p and the corestriction map is an isomorphism. This implies that f_* induces an isomorphism on cycle level.

□

In the next sections we will exploit the stratification

$$\Delta_{1,i} = \Delta_{1,i} \smallsetminus \Delta_{2,i} \sqcup \Delta_{2,i} \smallsetminus \Delta_{3,i} \sqcup \ldots \sqcup \Delta_{[i/2],i}$$

and the isomorphism

$$A^0_G(\Delta_{r,i} \smallsetminus \Delta_{r+1,i}) \simeq A^0_G((P^{i-2r} \smallsetminus \Delta_{1,i-2r}) \times P^r)$$

to inductively compute $A^0_G(\Delta_{1,i})$ and $A^0_G(P^i \smallsetminus \Delta_{1,i})$.

2 The invariants of $\mathscr{M}_2$

In this section we will compute the cohomological invariants of $\mathscr{M}_2$, assuming we are working over an algebraically closed field. Thanks to GL_2 being a special group, we will be able to do our computation by just looking at the chow groups mod p. The result is as follows:

Theorem 2.1. *Suppose the base field k_0 is algebraically closed, of characteristic different from $2,3$. Then the cohomological invariants of $\mathscr{M}_2$ are trivial if $p \notin \{2,5\}$, and:*

- *If $p = 2$, $\mathrm{Inv}(\mathscr{M}_2)$ is freely generated as a graded $\mathbb{F}_2$-module by 1 and elements x_1, x_2, x_3, x_4 respectively of degree $1, 2, 3, 4$.*
- *If $p = 5$, $\mathrm{Inv}(\mathscr{M}_2)$ is freely generated as a graded $\mathbb{F}_5$-module by 1 and an element x_1 of degree 1.*

A few last considerations on equivariant Chow rings are needed.

Lemma 2.2. *Let F be a vector bundle of rank 2 on a variety S smooth over k_0, let $P = P(F)$ be the projective bundle of lines in F, and Δ the image of the diagonal embedding $\delta : P \to P \times_S P$. Let x_1, x_2 in $\mathrm{A}^\bullet(P \times_S P)$ be the two pullbacks of the first Chern class of $\mathscr{O}_P(1)$, $c_1 \in \mathrm{A}^\bullet(P \times_S P)$ the pullback of the first Chern class of F. Then the class of Δ is $x_1 + x_2 + c_1$.*

Proof. This is [Vis96, 3.8]. □

Using the previous lemma we are able to compute the classes of $\Delta_{1,i}$ in $\mathrm{CH}^1_{GL_2}(P^i)$. Recall that the GL_2-equivariant Chow ring of $\mathbb{P}^i$ is generated by the Chern classes λ_1, λ_2 of the Hodge bundle and the first Chern class of $\mathscr{O}_{P^i}(1)$, which we will call t_i, and the only relation is a polynomial $f_i(t_i, \lambda_1, \lambda_2)$ of degree $i+1$ ([EG96, 3.2, prop.6] and the formula for projective bundles).

Proposition 2.3. *The class of $\Delta_{1,2}$ in $\mathrm{CH}^1_{GL_2}(P^2)$ is $2t_2 - 2\lambda_1$, The class of $\Delta_{1,4}$ in $\mathrm{CH}^1_{GL_2}(P^4)$ is $6t_4 - 12\lambda_1$, and the class of $\Delta_{1,6}$ in $\mathrm{CH}^1_{GL_2}(P^6)$ is $10t_6 - 30\lambda_1$.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} (P^1)^{(i-2)} \times P^1 & \xrightarrow{\ i\ } & (P^1)^i \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ P^{i-2} \times P^1 & \xrightarrow{\ \pi_{1,i}\ } & P^i \end{array}$$

The map ρ_1 is defined by $(f_1, \ldots, f_{i-2}, g) \to (f_1 \ldots f_{i-2}, g)$, the map ρ_2 is defined by $(f_1, \ldots, f_i) \to (f_1 \ldots f_i)$, and the map i is defined by $(f_1, \ldots, f_{i-2}, g) \to (f_1, \ldots, f_{i-2}, g, g)$. All the maps in the diagram are GL_2-equivariant, i is a closed immersion, ρ_1, ρ_2 are finite of degree respectively $(i-2)!$ and $i!$.

Now, the class of $\Delta_{1,i}$ is the image of 1 through $(\pi_{1,i})_*$. Following the left side of the diagram we obtain $[\Delta_{1,i}] = \frac{1}{(i-2)!}(\pi_{1,i} \circ \rho_1)^*(1)$. Consider now the right side of the diagram. The equivariant chow ring of $(P^1)^i$ is generated by all the different pullbacks of t_1, which we will call $x_1, \ldots, x_i$, plus λ_1 and λ_2. It is easy to check that the pullback of t_i is $x_1 + \ldots + x_i$, which by the projection formula, and by symmetry, implies that $\rho_*(x_j) = (i-1)!t_i$.

Using lemma 2.2, we see that $i_*(1) = x_i + x_{i-1} + \lambda_1$. Its image is $2(i-1)!t_i + i!\lambda_1$. By comparing the two formulas we obtain the statement of the proposition. □

We are ready to prove theorem 2.1. The proof will require a few steps.

Proof of Theorem 2.1. During this proof, we will often use the fact that by (5.3,5.17), if our base field is algebraically closed then the rings $\mathrm{A}^\bullet_{GL_2}(P(E_i))$ consist only of geometric elements, i.e. the map

$$\mathrm{CH}^\bullet_{GL_2}(P(E_i))/(p) \to \mathrm{A}^\bullet_{GL_2}(P(E_i))$$

is an isomorphism.

1. *If* $p = 2$ *then* $\mathrm{A}^0_{GL_2}(\Delta_{1,4}) = \langle 1, z_1 \rangle$, *where the degree of* z_1 *is* 1, *otherwise* $\mathrm{A}^0_{GL_2}(\Delta_{1,4}) = \langle 1 \rangle$.

   We begin by considering the exact sequence

$$0 \to \mathrm{A}^0_{GL_2}(P^2) \to \mathrm{A}^0(P^2 \smallsetminus \Delta_{1,2}) \xrightarrow{\partial} \mathrm{A}^0(\Delta_{1,2}) \xrightarrow{i_*} \mathrm{A}^1(P^2)$$

   As $i_*(1) = 2t_2 - 2\lambda_1$, i_* is zero for $p = 2$ and injective otherwise. In the first case, we have an element $z_1 \in \mathrm{A}^0(P^2 \smallsetminus \Delta_{1,2})$ of degree one, corresponding to an equation for $\Delta_{1,2}$, seen as an element of $k(P^2)^*/k(P^2)^{*2}$. In the second case, it is trivial.

   Now, if $p \neq 2$, then $\mathrm{A}^0((P^2 \smallsetminus \Delta_{1,2}) \times P^1) \simeq \mathrm{A}^0(\Delta_{1,4} \smallsetminus \Delta_{2,4})$ is trivial, implying the same for $\Delta_{1,4}$.

   If $p = 2$, we first consider the following commutative diagram with exact rows:

$$\begin{array}{ccc}
\mathrm{CH}_2^{GL_2}(\Delta_{1,2} \times P^1) & \xrightarrow{(\pi_1)_*} & \mathrm{CH}_2^{GL_2}(\Delta_{2,4}) \\
\Big\downarrow{\scriptstyle i_*} & & \Big\downarrow{\scriptstyle (i_2)_*} \\
\mathrm{CH}_2^{GL_2}(P^2 \times P^1) & \xrightarrow{\pi_*} & \mathrm{CH}_2^{GL_2}(\Delta_{1,4}) \\
\Big\downarrow{\scriptstyle j^*} & & \Big\downarrow{\scriptstyle j_2^*} \\
\mathrm{CH}_2^{GL_2}((P^2 \smallsetminus \Delta_{1,2}) \times P^1) & \xrightarrow{(\pi_2)_*} & \mathrm{CH}_2^{GL_2}(\Delta_{1,4} \smallsetminus \Delta_{2,4})
\end{array}$$

   The first horizontal map is multiplication by two, and the third one is an isomorphism. i_* and $(i_2)_*$ are injective because an irreducible effective divisor of a projective scheme cannot be numerically zero. We need to understand whether the class of $\Delta_{2,4}$ is divisible by two or not in the Picard group of $\Delta_{1,4}$.

   By the diagram we obtain the relation $2\,[\Delta_{2,4}] = 2\pi_*(t_2) + 2\pi_*\lambda_1$. Suppose now that $[\Delta_{2,4}] = 2\alpha$ for some $\alpha \in \mathrm{CH}_2^{GL_2}(\Delta_{1,4})$. Then 2α must belong to the kernel of j^*, but the only elements in the image of π_* whose double belongs to the kernel are the multiples of $\pi_*(t_2) + 2\pi_*\lambda_1$. This implies that 2α is equal to $2k\,[\Delta_{2,4}]$ for some k, and thus $(2k-1)\,[\Delta_{2,4}] = 0$, contradicting the injectivity of $(i_2)_*$.

   Consider now the exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(\Delta_{1,4}) \to \mathrm{A}^0_{GL_2}(\Delta_{1,4} \smallsetminus \Delta_{2,4}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{2,4}) \xrightarrow{i_*} \mathrm{A}^1_{GL_2}(\Delta_{1,4})$$

By proposition 1.3, $\Delta_{2,4}$ is universally homeomorphic to P^2, so that $\mathrm{A}^0(\Delta_{2,4})$ is concentrated in degree zero. Then the map i_* must be injective, and ∂ must be zero.

2. *We have* $\mathrm{A}^0_{GL_2}(\Delta_{2,6}) = \langle 1 \rangle$.

   As $\Delta_{2,6} \smallsetminus \Delta_{3,6}$ is universally homeomorphic to $(P^2 \smallsetminus \Delta_{1,2}) \times P^2$, for $p \neq 2$ the result is immediate. If $p = 2$ we consider the exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(\Delta_{2,6}) \to \mathrm{A}^0_{GL_2}(\Delta_{2,6} \smallsetminus \Delta_{3,6}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{3,6}) \xrightarrow{i_*} \mathrm{A}^1_{GL_2}(\Delta_{2,6})$$

   We know that $\mathrm{A}^0_{GL_2}(\Delta_{2,6} \smallsetminus \Delta_{3,6}) = \langle 1, x \rangle$ and x has degree one. In order to prove that x does not come from an element of $\mathrm{A}^0_{GL_2}(\Delta_{2,6})$, we just have to prove that i_* is not injective (recall that ∂ lowers degree by one.) As before, we consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{CH}_3^{GL_2}(\Delta_{1,2} \times P^2) & \xrightarrow{i_*} & \mathrm{CH}_3^{GL_2}(P^2 \times P^2) \\ \downarrow{\scriptstyle (\pi_1)_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{CH}_3^{GL_2}(\Delta_{3,6}) & \xrightarrow{(i_2)_*} & \mathrm{CH}_3^{GL_2}(\Delta_{2,6}) \end{array}$$

   The map $(\pi_1)_*$ is multiplication by three, and the image of 1 through i_* is $2(t_2 + \lambda_1)$, where t_2 is the pullback of $t_2 \in \mathrm{CH}^1_{GL_2}(P^2)$ through the first projection. Then we have $3\,[\Delta_{3,6}] = 2(t_2 + \lambda_1)$, implying $[\Delta_{3,6}] = 2(t_2 + \lambda_1 - [\Delta_{3,6}])$.

3. $\mathrm{A}^0_{GL_2}(\Delta_{1,6})$ *is equal to* $\langle 1, y_1, y_2 \rangle$, *with* $\deg(x_1) = 1, \deg(x_2) = 2$ *if* $p = 2$. *Otherwise,* $\mathrm{A}^0_{GL_2}(\Delta_{1,6})$ *is trivial.*

   Recall that $\Delta_{1,6} \smallsetminus \Delta_{2,6}$ is universally homeomorphic to $(P^4 \smallsetminus \Delta_{1,4}) \times P^1$, and $\mathrm{A}^0_{GL_2}((P^4 \smallsetminus \Delta_{1,4}) \times P^1) = \mathrm{A}^0_{GL_2}(P^4 \smallsetminus \Delta_{1,4})$. We can compute the latter by using the usual exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(P^4) \to \mathrm{A}^0_{GL_2}(P^4 \smallsetminus \Delta_{1,4}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{1,4}) \xrightarrow{i_*} \mathrm{A}^1_{GL_2}(P^4)$$

   The image of 1 through i_* is $6t_4 - 12\lambda_1$. For $p = 2$, $i_* = 0$ as $\mathrm{A}^1(P^4)$ is concentrated in degree zero and the class of $\Delta_{1,4}$ is 2-divisible. This implies that $\mathrm{A}^0_{GL_2}(P^4 \smallsetminus \Delta_{1,4})$ contains both an inverse image of x_1 through ∂, which we will name y_2 as well, and a new element of degree 1, y_1, corresponding to an equation for $\Delta_{1,4}$.

For $p = 3$ again i_* is zero, so that the element β in degree one appears. For $p \neq 2, 3$ the map i_* is injective, implying that $\partial = 0$, so $\mathrm{A}^0_{GL_2}(P^4 \smallsetminus \Delta_{1,4})$ is trivial. This proves the statement for $p \neq 2, 3$, as an open immersion induces an injective map on $\mathrm{A}^0_{GL_2}(-)$.

We now consider one more exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(\Delta_{1,6}) \to \mathrm{A}^0_{GL_2}(\Delta_{1,6} \smallsetminus \Delta_{2,6}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{2,6}) \xrightarrow{i_*} \mathrm{A}^1_{GL_2}(\Delta_{1,6})$$

We want to prove that i_* is injective for $p = 2$ and zero for $p = 3$. It suffices to prove that the class of $\Delta_{2,6}$ is divisible by three and not divisible by two in $\mathrm{CH}^{GL_2}_4(\Delta_{1,6})$. We proceed as before:

$$\begin{array}{ccc}
\mathrm{CH}^{GL_2}_4(\Delta_{1,4} \times P^1) & \xrightarrow{(\pi_1)_*} & \mathrm{CH}^{GL_2}_4(\Delta_{2,6}) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle (i_2)_*} \\
\mathrm{CH}^{GL_2}_4(P^4 \times P^1) & \xrightarrow{\pi_*} & \mathrm{CH}^{GL_2}_4(\Delta_{1,6}) \\
\downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j_2^*} \\
\mathrm{CH}^{GL_2}_4((P^4 \smallsetminus \Delta_{1,4}) \times P^1) & \xrightarrow{(\pi_2)_*} & \mathrm{CH}^{GL_4}_4(\Delta_{1,6} \smallsetminus \Delta_{2,6})
\end{array}$$

the map $(\pi_1)_*$ is just multiplication by two. This implies that $2\,[\Delta_{2,6}] = 6\pi_*(t_4 - 2\lambda_1)$ by the same reasoning as above, the only elements in the image of π_* whose double belongs to the kernel of j^* are multiples of $3\pi_*(t_4 - 2\lambda_1)$, and we easily get to the same contradiction as before. The class of $\Delta_{2,6}$ is divisible by three as $3(-2\pi_*(t_4 - 2\lambda_1) + [\Delta_{2,6}]) = [\Delta_{2,6}]$.

4. *If $p = 2$, then $\mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$ is equal to $\langle 1, x_1, x_2, x_3 \rangle$, where the degrees of x_1, x_2, x_3 are respectively $1, 2, 3$. If $p = 5$ then $\mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) = \langle 1, x_1 \rangle$ where x_1 is the class of an equation for $\Delta_{1,6}$. Otherwise, $\mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$ is trivial.*

This is instantly obtained by looking at the exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(P^6) \to \mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{1,6}) \xrightarrow{i_*} \mathrm{A}^1_{GL_2}(P^6)$$

And observing that the class of $\Delta_{1,6}$ in $\mathrm{A}^1_{GL_2}(P^6)$ is $10t_6 - 30\lambda_1$, so that i_* is zero for $p = 2, 5$ and injective otherwise.

This concludes the proof for $p \neq 2$ by proposition 1.2. For $p = 2$ we will have to look at the exact sequence induced by $P^6 \smallsetminus \Delta_{1,6} \xrightarrow{i} E$, where E is the total space of the GL_2-equivariant line bundle $\mathscr{O}(-1)\otimes\mathscr{D}^2$. We will do this in two steps. For the rest of the proof we assume $p=2$.

5. *In* $\mathrm{A}^\bullet_{GL_2}(P^4\smallsetminus\Delta_{1,4})$ *we have* $t_4y_2=0$, $t_4y_1\neq 0$.

We first take a look at the products in $\mathrm{A}^\bullet_{GL_2}(P^2\smallsetminus\Delta_{1,2})$. The second part of the localization exact sequence for $\Delta_{1,2}\to P^2$ reads:

$$0\to \mathrm{A}^1_{GL_2}(P^2)\to \mathrm{A}^1_{GL_2}(P^2\smallsetminus\Delta_{1,2})\xrightarrow{\partial}\mathrm{A}^1_{GL_2}(\Delta_{1,2})$$

We need to understand what t_2z_1 is. By the compatibility of Chern classes and boundary maps (5.2, chapter 2), we know that $\partial(t_2z_1)=\partial(c_1(\mathscr{O}_{P^2}(-1))(z_1))=c_1(i^*\mathscr{O}_{P^2}(-1))(\partial(z_1))=c_1(i^*\mathscr{O}_{P^2}(-1))(1)$. As the pullback of $\mathscr{O}_{P^2}(-1)$ through $P^1\xrightarrow{\pi_{1,2}}\Delta_{1,2}\to P^2$ is equal to $\mathscr{O}_{P^1}(-1)^2$, we see that $\partial(t_2z_1)=0$. Then t_2z_1 must be the image of some $\gamma\in\mathrm{A}^1_{GL_2}(P^2)$, but there are no element of positive degree in $A^\bullet_{GL_2}(P^2)$, so $t_2z_1=0$.

Consider now the elements $t_4\alpha, t_4\beta$ in $\mathrm{A}^1_{GL_2}(P^4\smallsetminus\Delta_{1,4})$. We will use the exact sequence

$$0\to \mathrm{A}^1_{GL_2}(P^4\smallsetminus\Delta_{2,4})\to \mathrm{A}^1_{GL_2}(P^4\smallsetminus\Delta_{1,4})\xrightarrow{\partial}\mathrm{A}^1_{GL_2}(\Delta_{1,4}\smallsetminus\Delta_{2,4})$$

Again by (5.2, chapter 2) we see that $\partial(t_4y_1)$ is equal to $c_1(i^*\mathscr{O}(-1))(\partial y_1)=c_1(i^*\mathscr{O}(-1))(1)$. We can now apply the projection formula to the map $\pi_{1,4}:(P^2\smallsetminus\Delta_{1,2})\times P^1\to\Delta_{1,4}$. As the pullback of $\mathscr{O}(-1)$ to $(P^2\smallsetminus\Delta_{1,2})\times P^1$ is equal to $\mathrm{pr}_1^*\mathscr{O}(-1)\otimes\mathrm{pr}_2^*\mathscr{O}(-1)^2$, we have $c_1(i^*\mathscr{O}(-1))(1)=(\pi_{1,4})_*t_2\neq 0$, implying that t_4y_1 must be nonzero.

On the other hand, applying the same reasoning to t_4y_2 we obtain $\partial(t_4y_2)=(\pi_{1,4})_*t_2z_1=0$. Then $t_4\alpha$ must belong to the image of $\mathrm{A}^1_{GL_2}(P^4\smallsetminus\Delta_{2,4})$, but looking at the exact sequence for $\Delta_{2,4}\to P^4$ we see that $\mathrm{A}^1_{GL_2}(P^4\smallsetminus\Delta_{2,4})$ contains elements of degree at most one, so t_4z_2 must be zero.

6. *Let* X *be the open subscheme of* $\mathbb{A}^7$ *consisting of nondegenerate forms of degree six. The graded* $\mathbb{F}_2$*-module* $\mathrm{A}^0_{GL_2}(X)$ *is generated by* 1 *and nonzero elements* x_1,x_2,x_3,x_4 *of respective degrees* $1,2,3,4$.

Let $E\xrightarrow{p}P^6\smallsetminus\Delta_{1,6}$ be the total space of the GL_2-equivariant line bundle $\mathscr{O}^{-1}\otimes\mathscr{D}^2$ on $P^6\smallsetminus\Delta_{1,6}$. Let $s:P^6\smallsetminus\Delta_{1,6}\to E$ be the zero section. We consider the exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(E) \xrightarrow{j^*} \mathrm{A}^0_{GL_2}(X) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) \xrightarrow{s_*} \mathrm{A}^1_{GL_2}(E)$$

If we identify the equivariant Chow groups with coefficients of E with those of $P^6 \smallsetminus \Delta_{1,6}$ using the isomorphism r defined in (page 36, chapter 2) we obtain the following exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) \xrightarrow{j^* \circ p^* \circ r} \mathrm{A}^0_{GL_2}(X) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) \xrightarrow{r \circ s_*} \mathrm{A}^1_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$$

The second map is equal to the pullback from $\mathrm{A}^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$ to $\mathrm{A}^0_{GL_2}(X)$, and the fourth map is equal to $c_1(\mathscr{O}(-1) \otimes \mathscr{D}^2)$ which is in turn equal to multiplication by t_6 as we are working modulo two. So the question boils down to whether $t_6 x_1, t_6 x_2$ and $t_6 x_3$ are zero or not in $\mathrm{A}^1_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$.

In analogy with the case of $P^4 \smallsetminus \Delta_{1,4}$, we consider the exact sequence

$$0 \to \mathrm{A}^1_{GL_2}(P^6 \smallsetminus \Delta_{2,6}) \to \mathrm{A}^1_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) \xrightarrow{\partial} \mathrm{A}^1_{GL_2}(\Delta_{1,6} \smallsetminus \Delta_{2,6})$$

Following the same reasoning as before we see that the boundaries of $t_6 x_1$ and $t_6 x_2$ cannot be zero as they are respectively equal to $(\pi_{1,6})_* t_4$ and $(\pi_{1,6})_* t_4 y_1$. The differential of $t_6 \alpha$ on the other hand is equal to $(\pi_{1,6})_* t_4 y_2 = 0$, and $\mathrm{A}^1_{GL_2}(P^6 \smallsetminus \Delta_{2,6})$ contains elements of degree at most one because $\mathrm{A}^0_{GL_2}(\Delta_{2,6}) = \langle 1 \rangle$, implying that it must be zero.

□

Remark 2.4. In this case we can understand almost completely the multiplicative structure of $\mathrm{Inv}^\bullet(\mathscr{M}_2)$. We have $\phi^2 = \phi\alpha = \phi\beta = \phi\gamma = 0$ as there are no elements of degree higher than ϕ, and similarly $\alpha^2 = \alpha\beta = \alpha\gamma = 0$ as these elements are pullbacks from $\mathrm{Inv}^\bullet([P^6 \smallsetminus \Delta_{1,6}/GL_2])$ and we can apply the same reasoning. The squares γ^2, β^2 are both zero, as the second is of degree 4 and there are no elements of degree 4 in $\mathrm{Inv}^\bullet([P^6 \smallsetminus \Delta_{1,6}/GL_2])$, and the first is represented by an element $\tilde{\gamma} \in \mathrm{H}^2(k(P^6)) = k(P^6) * /(k(P^6)^*)^2$ and squaring it we get the element $\{-1\}\tilde{\gamma} \in \mathrm{H}^2(k(P^6))$ which is zero as k contains a square root of -1. The product $\gamma\beta$ may be either equal to zero or to α.

In general we have no instruments to understand the multiplicative structure of $\mathrm{Inv}^\bullet(\mathscr{H}_g)$. The reason is that it is difficult to keep track of what our elements are when using the localization exact sequence, and in fact in most computations (that the author knows of) on classical cohomological invariants the multiplicative structure stems from an explicit *a priori* description of the invariants.

3 The invariants of $\mathscr{H}_g$, g even

In this section we will extend the result we obtained for $\mathscr{M}_2$ to all the stacks of hyperelliptic curves of even genus. Again we suppose that our base field k_0 is algebraically closed.

Proposition 3.1. *Let $p \neq 2$. If the class of $\Delta_{1,2i}$ is divisible by p in $\mathrm{CH}^1_{GL_2}(P^{2i})$ then $\mathrm{A}^0_{GL_2}(P^{2i} \smallsetminus \Delta_{1,2i})$ is generated by $\langle 1, \alpha \rangle$, where $\alpha \neq 0$ is the invariant in degree 1 corresponding to an equation for $\Delta_{1,2i}$. Otherwise $\mathrm{A}^0_{GL_2}(P^{2i} \smallsetminus \Delta_{1,2i})$ is trivial.*

Proof. We will proceed by induction on i, the base step being the groups we computed in the proof of theorem (2.1). Due to the localization exact sequence and proposition (1.2), the statement is equivalent to proving that for all i the group $\mathrm{A}^0_{GL_2}(\Delta_{1,2i})$ is concentrated in degree 0.

Consider the exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(\Delta_{1,2i}) \to \mathrm{A}^0_{GL_2}(\Delta_{1,2i} \smallsetminus \Delta_{2,2i}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{2,2i}) \xrightarrow{i_*} \mathrm{A}^1_{GL_2}(\Delta_{1,2i})$$

Due to the universal homeomorphism $\mathrm{A}^0_{GL_2}(P^{2i-2} \smallsetminus \Delta_{1,2i-2}) \times P^1 \to \Delta_{1,2i} \smallsetminus \Delta_{2,2i}$, the group $\mathrm{A}^0_{GL_2}(\Delta_{1,2i} \smallsetminus \Delta_{2,2i})$ is isomorphic to $\mathrm{A}^0_{GL_2}(P^{2i-2} \smallsetminus \Delta_{1,2i-2})$, so by the inductive hypothesis $\mathrm{A}^0_{GL_2}(\Delta_{1,2i} \smallsetminus \Delta_{2,2i})$ has at most a single nontrivial element α in degree one. If the group is trivial, we are done. Otherwise, we must show that the map i_* vanishes in degree zero. We know that the class of $\Delta_{1,2i-2}$ is divisible by p in $\mathrm{A}^1_{GL_2}(P^{2i-2})$. Consider the diagram:

$$\begin{array}{ccc}
CH^{GL_2}_{2i-2}(\Delta_{1,2i-2} \times P^1) & \xrightarrow{(\pi_1)_*} & CH^{GL_2}_{2i-2}(\Delta_{2,2i}) \\
\downarrow i_* & & \downarrow (i_2)_* \\
CH^{GL_2}_{2i-2}(P^{2i-2} \times P^1) & \xrightarrow{\pi_*} & CH^{GL_2}_{2i-2}(\Delta_{1,2i}) \\
\downarrow j^* & & \downarrow j_2^* \\
CH^{GL_2}_{2i-2}((P^{2i-2} \smallsetminus \Delta_{1,2i-2}) \times P^1) & \xrightarrow{(\pi_2)_*} & CH^{GL_2}_{2i-2}(\Delta_{1,2i} \smallsetminus \Delta_{2,2i})
\end{array}$$

As before, π_1 is multiplication by two, and we obtain that two times the class of $\Delta_{2,2i}$ is equal to the image of the class of $\Delta_{1,2i-2} \times P^1$. As the latter is by hypothesis divisible by p, there is an element ψ of $\mathrm{CH}^{GL_2}_{2i-2}(\Delta_{1,2i})$ such that $p\psi = 2\,[\Delta_{2,2i}]$. as p is odd, $\frac{p-1}{2}p\psi = p-1\,[\Delta_{2,2i}]$ so that $p([\Delta_{2,2i}] - \frac{p-1}{2}\psi) = [\Delta_{2,2i}]$. As $[\Delta_{2,2i}]$ is divisible by p, the map i_* must vanish in degree zero. $\square$

Remark 3.2. The result for $p \neq 2$ does not really require k_0 to be algebraically closed; it works in the same way by considering instead of a single element the $\mathrm{H}^\bullet(k_0)$-module it generates, and using the fact that all the maps we are considering are maps of $\mathrm{H}^\bullet(k_0)$-modules. This shows that the cohomological invariants of $[P^{2i} \smallsetminus \Delta_{1,2i}/GL_2]$ are generated at most by a single element of degree 1 for $p \neq 2$.

Proposition 3.3. *Let $p = 2$. If r is odd, the inclusion map $\Delta_{r,2i} \smallsetminus \Delta_{r+1,2i} \xrightarrow{j} \Delta_{r,2i}$ induces an isomorphism on $\mathrm{A}^0_{GL_2}$. If r is even, $\mathrm{A}^0_{GL_2}(\Delta_{r,2i})$ is trivial.*

Proof. As $\mathrm{A}^0_{GL_2}(\Delta_{r,2i})$ is isomorphic to $\mathrm{A}^0_{GL_2}(\Delta_{r,2i} \smallsetminus \Delta_{r+2,2i})$ (because $\Delta_{r+2,2i}$ has codimension two in $\Delta_{r,2i}$) we can compute it using the following exact sequence:

$$0 \to \mathrm{A}^0_{GL_2}(\Delta_{r,2i} \smallsetminus \Delta_{r+2,2i}) \to \mathrm{A}^0_{GL_2}(\Delta_{r,2i} \smallsetminus \Delta_{r+1,2i}) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(\Delta_{r+1,2i} \smallsetminus \Delta_{r+2,2i})$$

We want to prove that the kernel of ∂ is contained in degree zero when r is even, and that ∂ is zero when r is odd. The map $(P^{2i-2r} \smallsetminus \Delta_{2,2r}) \times P^r \xrightarrow{\pi} \Delta_{r,2i} \smallsetminus \Delta_{r+2,2i}$ yields the following commutative diagram with exact rows:

$$\begin{array}{ccc}
\mathrm{A}^0_{GL_2}((P^{2i-2r} \smallsetminus \Delta_{2,2i-2r}) \times P^r) & \xrightarrow{\pi_*} & \mathrm{A}^0_{GL_2}(\Delta_{r,2i} \smallsetminus \Delta_{r+2,2i}) \\
\downarrow & & \twoheaddownarrow \\
\mathrm{A}^0_{GL_2}((P^{2i-2r} \smallsetminus \Delta_{1,2i-2r}) \times P^r) & \overset{\pi_*}{\hookrightarrow\!\!\!\!\rightarrow} & \mathrm{A}^0_{GL_2}(\Delta_{r,2i} \smallsetminus \Delta_{r+1,2i}) \\
\downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial} \\
\mathrm{A}^0_{GL_2}((\Delta_{1,2i-2r} \smallsetminus \Delta_{2,2i-2r}) \times P^r) & \xrightarrow{\pi_*} & \mathrm{A}^0_{GL_2}(\Delta_{r+1,2i} \smallsetminus \Delta_{r+2,2i})
\end{array}$$

The second horizontal map is an isomorphism because π_* is a universal homeomorphism when restricted to $\Delta_{r,2i} \smallsetminus \Delta_{r+1,2i}$. The kernel of ∂_1 is contained in degree one because $\mathrm{A}^0_{GL_2}((P^{2i-2r} \smallsetminus \Delta_{2,2i-2r}) \times P^r$ is trivial. We claim that when r is even the third horizontal map is an isomorphism, implying that the kernel of ∂ must be contained in degree zero too, and when r is odd the third horizontal map is zero, so that ∂ must be zero too.

Let ψ be the map from $(P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2}) \times P^r \times P^1$ to $(P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2}) \times P^{r+1}$ sending (f,g,h) to (f,gh). We have a commutative diagram:

$$\begin{array}{ccc}
(P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2}) \times P^1 \times P^r & \xrightarrow{\pi_1} & (\Delta_{1,2i-2r} \smallsetminus \Delta_{2,2i-2r}) \times P^r \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \pi} \\
(P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2}) \times P^{r+1} & \xrightarrow{\pi_2} & \Delta_{r+1,2i} \smallsetminus \Delta_{r+2,2i}
\end{array}$$

Where π_1 and π_2 are defined respectively by $(f,g,h) \to (fg^2,h)$ and $(f,g) \to (fg^2)$. As π_1 and π_2 are universal homeomorphisms, if we prove that ψ_* is an isomorphism then π_* will be an isomorphism too, and if ψ_* is zero then π_* will be zero too. Consider this last diagram:

$$\begin{array}{ccc} (P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2}) \times P^r \times P^1 & \xrightarrow{p_1} & \\ \downarrow \psi & & \\ (P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2}) \times P^{r+1} & \xrightarrow{p_2} & P^{2i-2r-2} \smallsetminus \Delta_{1,2i-2r-2} \end{array}$$

The pullbacks of p_1 and p_2 are both isomorphism, implying that the pullback of ψ is an isomorphism. By the projection formula, $\psi_*(\psi^*\alpha) = \deg(\psi)\alpha$. Then as the degree of ψ is $r+1$, ψ_* is an isomorphism if r is even and zero if r is odd.

□

Remark 3.4. Again this result does not require any hypothesis on k_0. The result can be also proven by direct computation on the Chow groups mod 2, with the slight additional complication that the map $\mathrm{CH}^{GL_2}_{\bullet}(P^{2i-2r} \smallsetminus P^r) \xrightarrow{\pi_*} \mathrm{CH}^{GL_2}_{\bullet}(\Delta_{r,2i} \smallsetminus \Delta_{r+1,2i})$ is no longer surjective, as some points will only lift up to a purely inseparable extension.

Interestingly, the proposition above also gives us information on the Chow groups of $\Delta_{r,2i}$, as it implies that for the class of $\Delta_{r+1,2i}$ is 2-divisible in $\mathrm{CH}^{GL_2}_{\bullet}(\Delta_{r,2i})$ if and only if r is even.

From the next corollary on we will rely heavily on the algebraic closure of k_0, which is necessary to prove that the image of $i_* : A^0_{GL_2}(\Delta_{1,2i}) \to A^0_{GL_2}(P^{2i})$ is zero. In the next sections we will explore some ideas to get around this obstacle.

Corollary 3.5. *If $p = 2$, then $\mathrm{A}^0_{GL_2}(P^{2i} \smallsetminus \Delta_{1,2i}) = \langle 1, x_1, \ldots, x_i \rangle$, where the degree of x_i is i, and all the x_i are nonzero.*

Proof. We obtain the corollary immediately by induction using the last proposition and the localization exact sequence. □

Theorem 3.6. *Suppose our base field k_0 is algebraically closed, of characteristic different from $2, 3$, and let g be even. For $p = 2$, the cohomological invariants of $\mathscr{H}_g$ are generated as a graded $\mathbb{F}_2$-module by nonzero invariants $x_1, \ldots, x_{g+2}$, where the degree of x_i is i. If $p \neq 2$, then the cohomological invariants of $\mathscr{H}_g$ are nontrivial if and only if $2g+1$ is divisible by p. In this case they are generated by a single nonzero invariant in degree one.*

Proof. In the case $p \neq 2$ this is a direct consequence of lemma 1.2 and proposition 3.1. The explicit result for $p \neq 2$ can be obtained by looking at whether the class of $\Delta_{1,2g+2}$ is divisible by p in the equivariant Picard group of P^{2g+2}, which can be easily done using proposition 2.3.

If $p = 2$ we start from corollary 3.5. As we did at the end of the proof of theorem 2.1, we have to consider the exact sequence

$$0 \to \mathrm{A}^0_{GL_2}(P^{4g+2} \smallsetminus \Delta_{1,4g+2}) \xrightarrow{j^* \circ p^* \circ r} \mathrm{A}^0_{GL_2}(\mathbb{A}^{4g+3} \smallsetminus \Delta) \xrightarrow{\partial} \mathrm{A}^0_{GL_2}(P^{4g+2} \smallsetminus \Delta_{1,4g+2}) \xrightarrow{r \circ s_*} \mathrm{A}^1_{GL_2}(P^{2g+2} \smallsetminus \Delta_{1,2g+2})$$

So the question again boils down to understanding whether the products $t_{4g+2}x_j$ are zero or not. Suppose by induction that for $2i < 4g+2$ we know that $t_{2i}x_j \in \mathrm{A}^0_{GL_2}(P^{2i} \smallsetminus \Delta_{1,2i})$ is equal to zero if and only if $j = i$. We already know that for $i \leq 3$, giving us the base for the induction. Again, we can use the properties of Chern classes to see that, if we consider the exact sequence

$$0 \to \mathrm{A}^1_{GL_2}(P^{4g+2} \smallsetminus \Delta_{2,4g+2}) \to \mathrm{A}^1_{GL_2}(P^{4g+2} \smallsetminus \Delta_{1,4g+2}) \xrightarrow{\partial} \mathrm{A}^1_{GL_2}(\Delta_{1,4g+2} \smallsetminus \Delta_{2,4g+2})$$

Then the image of $t_{4g+2}x_j$ through ∂ is equal to $(\pi_{1,4g+2})_* t_{4g}x_{j-1}$, where we consider $x_0 = 1$. This immediately implies the thesis for $j < 2g+1$. As $\partial(t_{4g+2}x_{2g+1}) = 0$, it must come from an element of $\mathrm{A}^1_{GL_2}(P^{4g+2} \smallsetminus \Delta_{2,4g+2})$. The elements of $\mathrm{A}^1_{GL_2}(P^{4g+2} \smallsetminus \Delta_{2,4g+2})$ can have degrees only up to one plus the maximum degree of an element of $\mathrm{A}^0_{GL_2}(\Delta_{2,4g+2})$. Then by proposition 3.3 their degree is equal or lesser than one, and $t_{4g+2}x_{2g+1}$ must be zero, concluding the proof. □

4 The non algebraically closed case

In this section we obtain a partial result on the cohomological invariants of $\mathscr{M}_2$ for a general base field. This should give an idea of the inherent problems that arise when we have nontrivial elements of positive degree in our base rings. Note that this will happen even for an algebraically closed field if we are considering quotients by groups that are not special, making the development of techniques and ideas to treat these type of problems crucial for the future development of the theory.

Theorem 4.1. *Suppose that the characteristic of k_0 is different from $2, 3$.*

Let p be a prime different from 2. Then the cohomological invariants of $\mathscr{H}_g$ are nontrivial if and only if $4g+1$ is divisible by p. In this case they are freely generated by 1 and a single nonzero invariant in degree one.

Let p be equal to 2. Then the cohomological invariants of $\mathscr{M}_2$ are isomorphic as a $\mathrm{H}^\bullet(\mathrm{Spec}(k))$-module to a direct sum $M \oplus K$ where M is freely generated by 1 and elements x_1, x_2, x_3 of respective degrees $1, 2, 3$ and K is a submodule of $\mathrm{H}^\bullet(\mathrm{Spec}(k))$ [4].

The first statement is a direct consequence of remark (3.2) and proposition (1.2). The case $p = 2$ will require some work, and in the rest of the section we always work in this case. We begin by simplifying the last step:

Lemma 4.2. *The map $A^0_{GL_2}(\Delta_{1,6}) \to A^1_{GL_2}(P^6)$ is zero if and only if the map $A^0(\Delta_{1,6}) \to A^1(\Delta_{1,6})$ is zero.*

Proof. One arrow is trivial: the equivariant groups for P^6 map surjectively on the non-equivariant groups and the assignment is functorial, so if the equivariant map is trivial the same must be true for the non-equivariant map.

We now remove $\Delta_{2,6}$ from both sides, so that we are reduced to considering the map $(P^4 \smallsetminus \Delta_{1,4}) \times P^1 \to P^6 \smallsetminus \Delta_{2,6}$. All elements in $A^0_{GL_2}((P^4 \smallsetminus \Delta_{1,4}) \times P^1)$ are pullbacks through the first projection. An element $\alpha \in A^0_{GL_2}(P^4 \smallsetminus \Delta_{1,4})$ satisfies the equation $(t^5 + t^3\lambda_1^2)\alpha = 0$, where t is the first Chern class of $\mathscr{O}_{P^4}(-1)$ and λ_1 is the first Chern class of the determinant line bundle. The pullback $\alpha \in A^0((P^4 \smallsetminus \Delta_{1,4}) \times P^1)$ must then satisfy the same equation.

Note now that modulo two the pullback of $\mathscr{O}_{P^6}(-1)$ is equal to $\mathscr{O}_{P^4}(-1)$. As λ_1 is also the pullback of the corresponding equivariant line bundle on P^6, by the projection formula we see that the image of α must satisfy the same equation. As $i_*(\alpha)$ is an element of $A^1_{GL_2}(P^6)$ we can write $i_*(\alpha) = \lambda_1 \cdot a + t \cdot b$ with $a, b \in H^\bullet(\mathrm{spec}(k))$. Then we have $(t^5 + t^3\lambda_1^2)(\lambda_1 \cdot a + t \cdot b) = 0$ in $A^6_{GL_2}(P^6 \smallsetminus \Delta_{2,6})$.

Suppose that we know the result in the non-equivariant case, that is, we know that $b = 0$. We want to show that $(t^5 + t^3\lambda_1^2)\lambda_1 \cdot a$ belongs to the image of $A^4_{GL_2}(\Delta_{2,6})$ if and only if $a = 0$. Recall that $\Delta_{2,6}$ can be seen as the disjoint union of $(P^2 \smallsetminus \Delta_{1,2}) \times P^2$ and $\Delta_{3,6}$. We can divide the elements in $A^\bullet_{GL_2}(\Delta_{2,6})$ in three categories: those that come from $P^2 \times P^2$, those that come from $\Delta_{3,6}$ (which is universally homeomorphic to P^3) and the elements of $A^\bullet_{GL_2}((P^2 \smallsetminus \Delta_{1,2}) \times P^2)$ that are ramified on $\Delta_{1,2} \times P_2$ but unramified on $\Delta_{3,6}$.

Using the computations in [Vis96] we see that the first two images form the ideal $(t^6 + t^5\lambda_1 + t^4\lambda_1^2 + t^3\lambda_1^3)$. For the latter, the computation reduces to finding out the kernel of the pushforward $A^\bullet_{GL_2}(P^1 \times P^2) \to A^\bullet(P^3)$. Using the fact that the map is finite of degree 3 one sees that if we write $t = c_1(\mathscr{O}_{P^1}(-1)), s = c_1(\mathscr{O}_{P^2}(-1))$ the kernel is generated as a $A^\bullet_{GL_2}(\mathrm{Spec}(k))$-module by $1, s, st$. Then any element in codimension 4 belonging to the kernel of our pushforward can be written down as a sum $\lambda_1^2 a_1 + \lambda_2 a_2$, and the same must hold for any element in $A^4_{GL_2}(\Delta_{2,6})$ belonging to the third category. By the projection formula we can conclude that the image of $A^4_{GL_2}(\Delta_{2,6})$ must be contained in the ideal $(t^6 + t^5\lambda_1 + t^4\lambda_1^2 + t^3\lambda_1^3, \lambda_1^2, \lambda_2)$, which does not contain $(t^5 + t^3\lambda_1^2)\lambda_1 \cdot a$ unless $a = 0$. □

Of course the same trick will not work on the non-equivariant case, as the relation would be $t^6 \cdot a = 0$ and $\Delta_{2,6}$ contains rational points. We will have to dirty our hands and work at cycle level. Recall that the first Chern class of a line bundle L can be defined on cycles up to choosing a coordination for L.

Lemma 4.3. *Let $X \times \mathbb{A}^1$ be a trivial vector bundle with zero section σ and let τ be any coordination. Then $c_{1,\tau}(X \times \mathbb{A}^1) = r_\tau \circ \sigma_* : C^\bullet(X) \to C^\bullet(X)$ has the property that for every α there is β such that $r_\tau \circ \sigma_*(\alpha) = \mathrm{d}(\beta)$.*

Proof. We will proceed by induction on the length n of our coordination $\tau = (X_0 = X, X_1, \ldots, X_n = \emptyset)$. Recall that if we have constructed the map r_{τ_1} where τ_1 is the coordination restricted to X_1 we obtain the map r_τ by the formula:

$$r_\tau = \begin{pmatrix} r_{triv} & 0 \\ r_{\tau_1} \circ \partial^{X \smallsetminus X_1 \times \mathbb{A}^1}_{X_1 \times \mathbb{A}^1} \circ H_{triv} & r_{\tau_1} \end{pmatrix}$$

We first prove by induction that $r_\tau(\{t\}\pi^*(\alpha))$ is zero for all α, where $\mathbb{A}^1 = \mathrm{Spec}(k\,[t])$. This element is well defined as $\pi^*(\alpha)$ does not have any component lying on the zero section. When the coordination is trivial $r_\tau(\{t\}\pi^*(\alpha))$ is zero because $\{t\}\{-\frac{1}{t}\} = -\{t\}\{-t\} = 0$. In general, we have $H_{triv}(\{t\}\pi^*(\alpha)) = 0$ by direct computation in the same way that $H_{triv} \circ \pi^* = 0$, so the formula above allows us to conclude.

Consider now an element $\sigma_*(\alpha)$. If the coordination is trivial the result is trivially true. Consider now a general coordination τ, and suppose the result holds for τ_1. By direct computation we see that $H_{triv}(\sigma_*(\alpha)) = \{t\}\pi^*(\alpha)$. We separate the boundary map $\partial^{X \smallsetminus X_1 \times \mathbb{A}^1}_{X_1 \times \mathbb{A}^1}$ in two components:

$$\partial^{X \smallsetminus X_1 \times \mathbb{A}^1}_{X_1 \times \mathbb{A}^1} = \partial^{X \smallsetminus X_1 \times (\mathbb{A}^1 \smallsetminus \{0\})}_{X_1 \times (\mathbb{A}^1 \smallsetminus \{0\})} + \sigma^*_{X_1} \circ \partial^{X \smallsetminus X_1 \times \mathbb{A}^1}_{X_1 \times \mathbb{A}^1}$$

When computing the first component on the right we can consider t as an invertible element so that $\{t\}$ and $\partial^{X\smallsetminus X_1\times(\mathbb{A}^1\smallsetminus\{0\})}_{X_1\times(\mathbb{A}^1\smallsetminus\{0\})}$ anti-commute and we obtain an element in the form $\{t\}\pi^*_{X_1}(\beta)$ for some β, so that when we apply r_{τ_1} we get zero. The second component is contained in the zero section of X_1 so we can apply the inductive hypothesis. □

Lemma 4.4. *Let $E \to X$ be a line bundle that is isomorphic to $L\otimes W^{\otimes p}$ for some line bundles L, W. Let τ_L, τ_E be coordinations respectively for L and W, and consider the coordination for $\tau_L \cup \tau_W$ for E. Then for all α there is a β such that $c_{1,\tau_L\cup\tau_W}(E)(\alpha) = c_{1,\tau}(\alpha) + \mathrm{d}(\beta)$.*

Proof. It can be seen directly as in (5.8,chapter 2) that given a compatible choice of a trivialization and coordination for E the Chern class $c_{1,\tau}(E)$ is the sum of $c_{1,\tau}(X\times\mathbb{A}^1)$ and a function that is linear in the coordinate change elements $\alpha_{i,j} \in \mathscr{O}^*(U_i \times_X U_j)$. In the above situation the elements $\alpha_{i,j,E}$ satisfy $\alpha_{i,j,E} = \alpha_{i,j,L} \cdot \alpha^p_{i,j,W}$, allowing us to conclude. □

Proposition 4.5. *The map $A^0(\Delta_{1,6}) \to A^1(P^6)$ is zero.*

Proof. Given an element α in $A^0(\Delta_{1,6}) \to A^1(P^6)$ we know that it must come from $A^0((P^4\smallsetminus\Delta_{1,4}) \times P^1)$, which in turn comes from $\beta \in A^0(P^4\smallsetminus\Delta_{1,4})$. Consider a cycle $z \in C^0(P^4)$ mapping to α. Then if we consider $L = \mathscr{O}_{P^4}(-1)$, with the standard coordination τ given by the hyperplane at infinity, we have $c_{1,\tau}(L)^5(z) = 0$. Then the pullback L' of L to $P^4 \times P^1$ must satisfy $c_{1,\tau'}(L')^5(\beta) = 0$.

Consider now $E = \mathscr{O}_{P^6}(-1)$ with the standard coordination γ given again by the hyperplane at infinity. The pullback E' of E is $\mathscr{O}_{P^4}(-1)\otimes\mathscr{O}_{P^1}(-1)^2$, so we can see by the last lemma that $c_{1,\gamma'}(E')^5(\beta) = \mathrm{d}(\zeta)$ for some ζ in $C^4(P^1 \times P^4)$. The projection formula (on cycles) now tells us that the pushforward of β, which is an unramified element, must satisfy $c_1(E)^5(i_*(\beta)) = 0$ which implies $i_*(\beta) = 0$. □

Proposition 4.6. *The pullback $A^0_{GL_2}(\Delta_{1,2i}) \to A^0_{GL_2}(\Delta_{1,2i}\smallsetminus\Delta_{2,2i})$ is an isomorphism.*

Proof. This is remark (3.4). □

Proof of Theorem 4.1. 1. The maps

$$A^0_{GL_2}(\Delta_{1,2}) \to A^1_{GL_2}(P^2), \quad A^0_{GL_2}(\Delta_{1,4}) \to A^1_{GL_2}(P^4)$$

are both zero. The first statement is due to the projection formula. To check the second, note that by the previous points we have that $A^0_{GL_2}(\Delta_{1,4}) = \langle 1, \alpha\rangle$ as an $H^*(\mathrm{Spec}(k))$-module, where α is an element of degree 1. Moreover, as before we see that if we call c_1 the pullback of first Chern class of $\mathscr{O}_{P^4}(-1)$ we have $\partial(c_1 \cdot \alpha) = 0$, and consequently by the projection formula $c_1(\mathscr{O}_{P^4}(-1)(i_*\alpha)) = 0$, which by the structure of the Chow groups with coefficient of a projective bundle implies $i_*\alpha = 0$.

2. The points above and the preliminary results we have proven in the rest of this section easily imply that $A^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$ is freely generated as an $H^*(\mathrm{Spec}(k))$-module by 1 and elements x_1, x_2, x_3 of degree respectively $1, 2, 3$. All that is left to understand is the kernel of $c_1(\mathscr{O}_{P^6}(-1)) : A^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6}) \to A^0_{GL_2}(P^6 \smallsetminus \Delta_{1,6})$. We can proceed as in the previous sections to prove by induction that the map is injective on the submodule generated by $1, x_1, x_2$. Unfortunately the reasoning we used before to prove that γ must belong to the kernel of $c_1(\mathscr{O}_{P^6}(-1))$ no longer works, as it relied heavily on the algebraic closure of k, so we have to add the unspecified module K to our final result.

□

5 Some equivariant Chow groups with coefficients

In this section we compute some equivariant Chow groups with coefficients leading to $A^\bullet_{SO_3}(\mathrm{spec}(k_0))$, which we will use to compute the cohomological invariants of $\mathscr{H}_3$ using the isomorphism $SO_3 \simeq PGL_2$.

The computation has some interest by itself, and it does not require much effort to extend it to Chow groups with coefficients in Milnor's K-theory. We begin by computing the μ_p-equivariant Chow ring with coefficients of a point.

Proposition 5.1. *Let k be a field and q be a prime different from the characteristic of k.*

- *If M is Milnor's K-theory, then $A^\bullet_{\mu_q}(\mathrm{Spec}(k))$ is equal to ${M(\mathrm{Spec}(k))\,[\xi]}/{q\xi}$. Here ξ is the first Chern class of the standard one-dimensional representation of μ_n.*
- *If $M = \mathrm{H}^\bullet$ and $q \neq p$, then $A^\bullet_{\mu_q}(\mathrm{Spec}(k))$ is equal to $\mathrm{H}^\bullet(\mathrm{Spec}(k))$.*
- *If $M = \mathrm{H}^\bullet$ and $p = q$, then $A^\bullet_{\mu_q}(\mathrm{Spec}(k))$ is equal to $\mathrm{H}^\bullet(\mathrm{Spec}(k))\,[t, \xi]$. Here t is an element in codimension 0 and degree one, corresponding to a generator for the cohomological invariants of μ_q.*

Proof. We consider the action of μ_q on G_m induced by the inclusion. This action extends linearly to $\mathbb{A}^1_k$. Then there is a long exact sequence:

$$0 \to A^0_{\mu_q}(\mathbb{A}^1_k) \to A^0_{\mu_q}(G_m) \xrightarrow{\partial} A^0_{\mu_q}(\mathrm{Spec}(k)) \xrightarrow{c_1}\to A^1_{\mu_q}(\mathbb{A}^1_k) \to \dots$$

Using the retraction r described in [Ros96, section 9] we identify $A^{\bullet}_{\mu_q}(\mathbb{A}^1_k)$ with $A^{\bullet}_{\mu_q}(\mathrm{Spec}(k))$ and consequently the inclusion pushforward with the first Chern class for the equivariant vector bundle $\mathbb{A}^1_k \to \mathrm{Spec}(k)$. Note now that $[G_m/\mu_q] \simeq G_m$, so that $A^{\bullet}_{\mu_q}(G_m) \xrightarrow{\partial}= M(\mathrm{Spec}(k)) \oplus t \cdot M(\mathrm{Spec}(k))$, where t is an element in codimension zero and degree one. The differential of this element at the origin is equal to p. The computation immediately follows. □

The reasoning works the same for an algebraic space being acted on trivially by μ_q.

Lemma 5.2. *Let X be an algebraic space over a field k, and let μ_q act trivially on it. Then $A^{\bullet}_{\mu_q}(X) = A^{\bullet}(X) \otimes_{M(\mathrm{Spec}(k))} A^{\bullet}_{\mu_q}(\mathrm{Spec}(k))$.*

Proof. We consider again the exact sequence:

$$0 \to A^0_{\mu_q}(X \times \mathbb{A}^1) \xrightarrow{j^*} A^0_{\mu_q}(X \times G_m) \xrightarrow{\partial} A^0_{\mu_q}(X) \xrightarrow{c_1}\to A^1_{\mu_q}(X \times \mathbb{A}^1) \to \dots$$

As before, the quotient $[(X \times G_m)/\mu_q]$ is isomorphic to $X \times G_m$, so that for its Chow groups with coefficients the formula $A^i_{\mu_q}(X \times G_m) = A^i(X) \oplus t \cdot A^i(X)$ holds.

As the first component comes from the pullback through $X \times G_m \to X$ and this map factors through $[(X \times \mathbb{A}^1)/\mu_q]$ we see that the first component always belongs to the image of j^*, and given an element $\alpha \cdot t$ in the second component its image through the boundary map ∂ is equal to q times α. This gives us a complete understanding of the exact sequence, allowing us to conclude.

□

With the next proposition we compute the equivariant Chow ring $A^{\bullet}_{O_n}(\mathrm{Spec}(k))$ for $n = 2, 3$. This should serve as an example of how the Chow groups with coefficients can start behaving wildly even for well known objects, as elements of positive degree with no clear geometric or cohomological description appear almost immediately.

We will follow the method in [VM06, 4.1]. First we need a few more lemmas, which are by themselves interesting facts about the equivariant approach.

Lemma 5.3. *Let G be a linear algebraic group, acting on an algebraic space X smooth over k_0, and let H be a normal subgroup of G. Suppose the action of H on X is free with quotient X/H. Then there is a canonical isomorphism*

$$A^{\bullet}_{G}(X) \simeq A^{\bullet}_{G/H}(X/H)$$

.

Proof. The proof in [VM06][2.1] works without any change. □

Lemma 5.4. *Let H be a linear algebraic group with an isomorphism $\phi : H \simeq \mathbb{A}^n_k$ of varieties such that for any field extension $k' \supseteq k$ and any element $h \in H(k')$ the automorphism of $\mathbb{A}^n_k$ corresponding through ϕ to the action of h on H_k by left multiplication is affine. Furthermore, let G be a linear algebraic group acting on H via group automorphisms, corresponding to linear automorphisms of $\mathbb{A}^n_k$ under ϕ.*

If G acts on an algebraic space X smooth over k_0, form the semidirect product $G \rtimes H$ and let it act on X via the projection $G \rtimes H \to G$. Then the homomorphism

$$A^{\bullet}_{G}(X) \to A^{\bullet}_{G\rtimes H}(X)$$

induced by the projection $G \rtimes H \to G$ is an isomorphism.

Proof. Again the argument used in [VM06, 2.3] works for any equivariant theory defined as in [EG96]. □

Proposition 5.5. *Let M be equal to Milnor's K-theory. Let $R_{n,k}$ be the tensor product of the ordinary O_n-equivariant Chow groups of the spectrum of a field k with the field's K-theory, that is*

$$R_{n,k} = {}^{M(\mathrm{Spec}(k))\,[c_1,\ldots,c_n]}\big/_{(2c_i)_{(i\ odd)}}$$

Then

$$A^{\bullet}_{O_2}(\mathrm{Spec}(k)) = R_{2,k} \oplus R_{2,k}\tau_{1,1}$$

where $\tau_{1,1}$ is an element in codimension and degree one, with $2\tau_{1,1} = 0$. For $n = 3$ we have

$$A^{\bullet}_{O_3}(\mathrm{Spec}(k)) = R_{3,k} \oplus R_{3,k}\tau_{1,1} \oplus R_{3,k}\tau_{1,2}$$

where $\tau_{1,1}$ and $\tau_{1,2}$ are respectively of codimension and degree $(1,1)$ and $(1,2)$, and both are of 2-torsion.

Let M be equal to Galois Cohomology with coefficients in $\mathbb{F}_2$. Recall that $A^0_{O_n}(\mathrm{Spec}(k))$ is isomorphic the ring of cohomological invariants of O_n, which is generated as a $M(\mathrm{Spec}(k))$-algebra by the Steifel Whitney classes $1 = w_o, w_1, \dots, w_n$, where w_i has degree i. Let $c_1, \dots, c_n$ be the Chern classes of the standard representation of O_n. Then for $n = 2, 3$

$$A^\bullet_{O_n}(\mathrm{Spec}(k)) = A^0_{O_n}(\mathrm{Spec}(k))\,[c_1, \dots, c_n] \oplus M(\mathrm{Spec}(k))\,[c_1, \dots, c_n]\,\tau_{1,1}$$

Where again $\tau_{1,1}$ is an element of codimension and degree $(1,1)$.

Let M be equal to Galois Cohomology with coefficients in $\mathbb{F}_p$, with $p \neq 2$. Then $A^\bullet_{PGL_2}(\mathrm{Spec}(k))$ is equal to the tensor product of $M(k)$ with the ordinary equivariant Chow ring.

Proof. We will just have to adjust the original argument from [VM06, 4.1]. Consider the standard n-dimensional representation V of O_n. We want to compute $A^\bullet_{O_n}(V) = A^\bullet_{O_n}(\mathrm{Spec}(k))$. Let q be standard quadratic form being fixed by O_n. We will stratify V as the union of $B = \{q \neq 0\}, C = \{q = 0\} \smallsetminus \{0\}$ and the origin $\{0\}$.

First, the map $B \to G_m$ can be trivialized by passing to an étale covering $\tilde{B}$, with an action of μ_2 such that $\tilde{B}/\mu_2 = B$. If we call Q the locus where $q = 1$, then $\tilde{B}$ is isomorphic to $Q \times G_m$, the action of μ_2 is the multiplication on the second component and the action of O_n is the action on the first component. Following lemma (5.2) we see that $A^\bullet_{O_n}(B) = A^\bullet_{O_n \times \mu_2}(\tilde{B}) = A^\bullet_{O_n}(Q) \oplus A^\bullet_{O_n}(Q)t$, where t is an element in codimension 0 and degree 1.

Using lemmas (5.3,5.4) one sees that $A^\bullet_{O_n}(Q) = A^\bullet_{O_{n-1}}(\mathrm{Spec}(k))$ and $A^\bullet_{O_n}(C) = A^\bullet_{O_{n-2}}(\mathrm{Spec}(k))$. As we know the rings $A^\bullet_{O_n}(\mathrm{Spec}(k))$ for $n = 0, 1$, all that remains is to understand the long exact sequences coming from the equivariant inclusions $C \to V \smallsetminus \{0\}$ and $\{0\} \to V$.

For $N = 2$ we know that the ring $A^{O_2}(C)$ is equal to $M(\mathrm{Spec}(k))$ and that it must map to zero. This forces the differential of the element t to be equal to 1. As the map $A^\bullet_{O_2}(V \smallsetminus \{0\}) \to A^\bullet_{O_2}(B)$ is injective, we have $A^\bullet_{O_2}(V \smallsetminus \{0\}) = A^\bullet_{O_1 = \mu_2}(\mathrm{Spec}(k)) \oplus M(\mathrm{Spec}(k))\,[c_1]\,\tau 1, 1$. We can then conclude by observing that the map $A^\bullet_{O_2}(V) \to A^\bullet O_2(V \smallsetminus \{0\})$ is a map of rings and it is surjective in codimension 0 and in degree 0 for all codimensions; we can see that $\tau_{1,1}$ must be in the image as the second Chern class c_2 is injective i degree zero.

For $n = 3$, it suffices to do the same calculations knowing that the map $A^\bullet(C) \to A^\bullet(V \smallsetminus \{0\})$ must again be 0 (this is obvious in the case of K-theory, and can be seen for Galois cohomology as the pullback of w_1 to C corresponds to $w_1 \in A^0_{O_1}(\mathrm{Spec}(k))$). For the second long exact sequence we reason as above, except that in the case of K-theory the class c_3 has a kernel corresponding to the ideal (2), which means that there must be an element in codimension 2 and degree 1 mapping to $2 \in A^0_{O_3}(\mathrm{Spec}(k))$. We conclude by seeing that the choice of such element is forced. □

Corollary 5.6. *Let M be equal to Galois cohomology with coefficients in $\mathbb{F}_2$. The equivariant Chow ring with coefficient $A^\bullet_{SO_3}(\mathrm{Spec}(k))$ is isomorphic to*

$$A^0_{S_3}(\mathrm{Spec}(k))\,[c_3, c_3] \oplus M(\mathrm{Spec}(k))\,[c_2, c_3]\,\tau_{1,1}$$

Let M be equal to Galois cohomology with coefficients in $\mathbb{F}_2$. The equivariant Chow ring with coefficient $A^\bullet_{SO_3}(\mathrm{Spec}(k))$ is isomorphic to to the tensor product of $M(k)$ with the ordinary equivariant Chow ring.

Proof. It suffices to use the fact that $O_3 = \mu_2 \times SO_3$ and apply (5.2). □

6 The invariants of $\mathscr{H}_3$

In this section we will compute the cohomological invariants of the stack $\mathscr{H}_3$ of hyperelliptic curves of genus three over an algebraically closed field.

Recall that the presentation of $\mathscr{H}_3$ is obtained by considering $\mathbb{A}^{4g+1}$ as the space of all binary forms of degree $4g$, removing the subset Δ of binary forms with multiple roots and taking the $[(\mathbb{A}^{4g+1} \smallsetminus \Delta)/PGL_2 \times G_m]$, where the action of $PGL_2 \times G_m$ is given by $([A], \alpha)(f)(x) = \mathrm{Det}(A)^{2g}\alpha^{-2} f(A^{-1}(x))$.

There are various differences form the previous cases. First, PGL_2 is not special, and its Chow groups with coefficients have multiple elements in positive degree when $p = 2$:

Proposition 6.1. *Let p be equal to 2, and $M = \mathrm{H}^\bullet$. Then $A^\bullet_{PGL_2}(\mathrm{Spec}(k))$ is freely generated as a module over $\mathrm{CH}^\bullet_{PGL_2}(\mathrm{Spec}(k)) \otimes \mathrm{H}^\bullet(k)$ by the cohomological invariant v_2 and an element τ in degree and codimension $1, 1$.*

If $p \neq 2$, then $A^\bullet_{PGL_2}(\mathrm{Spec}(k))$ is equal to $\mathrm{CH}^\bullet_{PGL_2}(\mathrm{Spec}(k)) \otimes \mathrm{H}^\bullet(k)$.

Proof. As PGL_2 is isomorphic to SO_3, we can just apply (5.6). □

The second difference is that the action of PGL_2 on P^1 does not come from a linear action on the space of degree one forms. This is true in general whenever we are having PGL_2 act on a projective space of odd dimension. The following proposition describes the ring $A^{\bullet}_{PGL_2}(P^1)$.

Proposition 6.2. *The kernel of the map* $\pi^* : A^{\bullet}_{PGL_2}(\mathrm{Spec}(k)) \to A^{\bullet}_{PGL_2}(P^1)$ *is generated by* w_2, c_3, τ, *and* $A^{\bullet}_{PGL_2}(P^1) = {}^{\mathrm{Im}(\pi^*)\,[t]}\!/_{t^2 + c_2}$.

Proof. This can be proven exactly as in [FV11, 5.1]. □

We begin by proving the following lemma:

Lemma 6.3. *1. Suppose that the pullback*

$$A^0_{PGL_2}(P^{n-4} \smallsetminus \Delta_{1,n-4}) \to A^0_{PGL_2}((P^{n-4} \smallsetminus \Delta_{1,n-4}) \times P^1)$$

is surjective. Then the pullback

$$A^0(\Delta_{1,n}) \to A^0(\Delta_{1,n} \smallsetminus \Delta_{2,n})$$

is an isomorphism.

2. Suppose that the pullback

$$A^0_{PGL_2}(P^{n-6} \smallsetminus \Delta_{1,n-6-2i}) \to A^0_{PGL_2}((P^{n-6-2i} \smallsetminus \Delta_{1,n-6-2i}) \times P^1)$$

is surjective. Then the map

$$A^0_{PGL_2}(\mathrm{Spec}(k)) \to A^0_{PGL_2}(\Delta_{2+2i,n})$$

is surjective.

3. Suppose that the above holds and that the pushforward

$$A^0_{PGL_2}(\Delta_{1,n}) \to A^1_{PGL_2}(P^n)$$

is zero. Then the pullback

$$A^0_{PGL_2}(P^n \smallsetminus \Delta_{1,n}) \to A^0_{PGL_2}((P^n \smallsetminus \Delta_{1,n}) \times P^1)$$

is surjective and its kernel is generated by w_2, *the second Stiefel Whitney class coming from* $\mathrm{Inv}(PGL_2)$.

Proof. First, note that given a PGL_2-equivariant space X, while $X \times P^1 \to X$ is not the projectivization of an equivariant vector bundle, $X \times P^1 \times P^1 \to X \times P^1$ is, and so the pullback through the second map is an isomorphism in codimension zero.

Using this we see that we can apply the reasoning (3.3) word by word to prove the first point.

The elements of positive degree in $A^0_{PGL_2}((P^n \smallsetminus \Delta_{1,n}) \times P^1)$ are completely determined by their image through the boundary map

$$A^0_{PGL_2}((P^n \smallsetminus \Delta_{1,n}) \times P^1) \xrightarrow{\partial} A^0_{PGL_2}(\Delta_{1,n} \times P^1)$$

.

Consider now the mapping $\Delta_{1,n} \times P^1 \to \Delta_{1,n}$. If we remove $\Delta_{2,n}$ and its inverse image we obtain a pullback

$$A^0_{PGL_2}((P^{n-2} \smallsetminus \Delta_{1,n-2}) \times P^1) = A^0_{PGL_2}(\Delta_{1,n} \smallsetminus \Delta_{2,n}) \to A^0_{PGL_2}((\Delta_{1,n} \smallsetminus \Delta_{2,n}) \times P^1) = A^0_{PGL_2}((P^{n-2} \smallsetminus \Delta_{1,n-2}) \times P^1 \times P^1)$$

Then by condition 1 we know that $A^0_{PGL_2}(\Delta_{1,n})$ surjects over $A^0_{PGL_2}(\Delta_{1,n} \times P^1)$ and the fact that the map $A^0_{PGL_2}(\Delta_{1,n}) \to A^1_{PGL_2}(P^n$ is zero shows that for every element α in the first group there is an element in $A^0_{PGL_2}(P^n \smallsetminus \Delta_{1,n})$ whose boundary is exactly α. The compatibility of pullback and boundary maps and the surjectivity of the map above allow us to conclude. The description of the kernel stems from the fact that it must be generated by elements that are unramified on $\Delta_{1,n}$. □

The lemma almost provides an inductive step, as its conclusions provide all of its hypotheses except for the requirement that the pushforwards $A^0_{PGL_2}(\Delta_{1,n}) \to A^1_{PGL_2}(P^n)$ are zero. The following proposition gives us some information on the annihilator of the image of these pushforwards.

We introduce some elements of $A^\bullet_{PGL_2}(P^i)$:

$$f_n = \begin{cases} t_i^{n+4/4}(t_i^3 + c_2 t_i + c_3)^{n/4}, & \text{if } n \text{ is divisible by } 4 \\ t_i^{n-2/4}(t^3 + c_2 t_i + c_3)^{n+2/4}, & \text{if } n \text{ is not} \end{cases}$$

We have $A^\bullet_{PGL_2}(P^i) = A^\bullet_{PGL_2}(\mathrm{Spec}(k_0))/(f_i)$ by [FV11, 6.1] and the projective bundle formula.

Lemma 6.4. *Suppose that $p = 2$. Then the class of c_3 is zero in $A^\bullet_{PGL_2}(P^i)$ if and only if i is odd.*

Proof. If i is even then P^i is the projectivized of a representation of PGL_2 and the projective bundle formula allows us to conclude immediately. If i is odd we just have apply the projection formula to the equivariant map $P^1 \times P^{i-1} \to P^i$. $\square$

Proposition 6.5. *Let i be an even positive integer, and let α be an element of $A^0_{PGL_2}(\Delta_{1,i})$. Then:*

- *If i is divisible by 4, the image of α in $A^{\bullet}_{PGL_2}(P^i)$ is annihilated by $c_3^{i/4} f_{i-4} \dots f_4 t$.*
- *If i is not divisible by 4, the image of α in $A^{\bullet}_{PGL_2}(P^i)$ is annihilated by $c_3^{i+2/4} f_{i-4} \dots f_4$.*

Proof. Consider the map $\Delta_{1,i} \smallsetminus \Delta_{2,i} \xrightarrow{i} P^i \smallsetminus \Delta_{2,i}$. As $\Delta_{1,i} \smallsetminus \Delta_{2,i}$ is universally homeomorphic to $P^{i-2} \smallsetminus \Delta_{i-2,1} \times P^1$ we know by (6.2) that the pullback of c_3 through i must be zero. This shows that $c_3 i_* \alpha = 0$. As we already know that $c_3 i_* \alpha$ belongs to $A^1_{PGL_2}(P^i)$ it must belong to the kernel of $A^1_{PGL_2}(P^i) \to P^i \smallsetminus \Delta_{2,i}$, which is the image of $A^{\bullet}_{PGL_2}(\Delta_{2,i})$. Let β be a preimage of $c_3 i_* \alpha$.

Consider now $\beta \in A^2(\Delta_{2,i})$, and let β' be the pullback of β to $\Delta_{i,2} \smallsetminus \Delta_{i,3}$. We can see β' as an element of $A^2_{PGL_2}((P^{i-4} \smallsetminus \Delta_{3,i}) \times P^2)$. we know that in this ring the equation $f_{i-4}(t_{i-4}, c_2, c_3) = 0$ holds and as we are working mod 2 the pullback of $t_i \in A^1_{PGL_2}(P^i)$ is equal to $t_{i-4} \in A^1_{PGL_2}(P^{i-4})$ we see that the pullback of $f_{i-4}(t_i, c_2, c_3)$ is exactly $f_{i-4}(t_{i-4}, c_2, c_3) = 0$, implying that $f_{i-4}(t_i, c_2, c_3) i_* \beta' = 0$ in $A^{\bullet}_{PGL_2}(P^i \smallsetminus \Delta_{2,i}$. As before, this proves that $c_3 f_{i-4} i_* \alpha$ belongs to the image of $A^{\bullet}_{PGL_2}(\Delta_{3,i})$.

We can clearly repeat this reasoning inductively to move from $\Delta_{r,i}$ to $\Delta_{r+1,i}$, multiplying by c_3 and applying (6.4) if r is odd, and multiplying by f_{i-2r} is r is even. The last thing to note is that when $r = i/2$ the process end and we obtain 0, either multiplying by $f_0 = t$ if i is divisible by 4 or by c_3 otherwise. $\square$

Corollary 6.6. *Suppose that $p = 2$. Then the cohomological invariants of $[P^8/PGL_2]$ are freely generated as a $\mathrm{H}^{\bullet}(k_0)$-module by 1 and nonzero elements x_1, x_2, w_2, x_3, x_4, where the degree of x_i is i and w_2 is the second Stiefel-Whitney class coming from the cohomological invariants of PGL_2.*

If $p \neq 2$, then the cohomological invariants of $[P^i/PGL_2]$ are trivial unless p divides $i-1$, in which case they are generated as a $\mathrm{H}^{\bullet}(k_0)$-module by 1 and a single nonzero invariant of degree 1.

Proof. For $p = 2$, the proposition above shows that the maps $i_* : A^0_{PGL_2}(\Delta_{1,i}) \to A^1_{PGL_2}(P^i)$ are zero for $i \leq 8$, as the polynomial killing the image of i_* is not divisible by f_i. Then we can apply (6.3) repeatedly to obtain the result in the same way as we did for the g even case.

The case $p \neq 2$ can be proven exactly as in (3.1). □

Note that the reasoning above does not work for any $i > 8$ when $p = 2$.

Theorem 6.7. *Suppose that $p = 2$ and k_0 is algebraically closed. Then the cohomological invariants of $\mathscr{H}_3$ are freely generated as a $\mathrm{H}^\bullet(k_0)$-module by 1 and nonzero elements $x_1, x_2, w_2, x_3, x_4, x_5$, where the degree of x_i is i and w_2 is the second Stiefel-Whitney class coming from the cohomological invariants of PGL_2.*

In general, for $p = 2$ the cohomological invariants of $\mathscr{H}_3$ are a direct sum $M \oplus K$, where K is a submodule of $\mathrm{H}^\bullet(k_0)$ [5] and M is generated as a $\mathrm{H}^\bullet(k_0)$-module by 1 and nonzero elements x_1, x_2, w_2, x_3, x_4, where the degree of x_i is i and w_2 is the second Stiefel-Whitney class coming from the cohomological invariants of PGL_2.

If $p \neq 2$ for all odd g the cohomological invariants of $\mathscr{H}_g$ are trivial unless p divides $2g+1$, in which case they are generated as a $\mathrm{H}^\bullet(k_0)$-module by 1 and a single nonzero invariant of degree 1.

Proof. The case $p \neq 2$ is immediate from the previous corollary and lemma 1.2. For the rest of the proof we will have $p = 2$.

First, we observe that as G_m acts trivially on $[P^8 \smallsetminus \Delta_{1,8}/PGL_2]$ the map $[P^8 \smallsetminus \Delta_{1,8}/PGL_2] \to [P^8 \smallsetminus \Delta_{1,8}/PGL_2 \times G_m]$ induces an isomorphism on cohomological invariants.

We need to understand whether the G_m-torsor

$$\mathscr{H}_3 \to \left[P^8 \smallsetminus \Delta_{1,8}/PGL_2 \times G_m\right]$$

generates any new cohomological invariant.

This amounts to understanding the kernel of the first Chern class of the associated line bundle on $[P^8 \smallsetminus \Delta_{1,8}/PGL_2 \times G_m]$, and as before for $p = 2$ this is just the first Chern class of $\mathscr{O}_1$ [FV11, 3.2].

We can follow the same reasoning we used in proving the result for $\mathscr{M}_2$. For $x_1, \ldots, x_3$ we can inductively show that they can not be annihilated by t_8 as the boundary $\partial(t_8 x_i)$ is not zero. This is sufficient to prove the result for a general field.

Even if k_0 is algebraically closed, the matter is a bit more complicated than usual for x_4 as there are elements of positive degree in $A^0_{PGL_2}(\Delta_{2,i})$ coming from the Chow ring with coefficients of $BPGL_2$. To get around this problem, we make the following consideration. There are no elements of degree 4 in $A^0_{PGL_2}(P^8)$, so $t_8 x_4$ is zero if and only if its boundary $\partial(t_8 x_4)$ is zero in $\Delta_{1,8}$. As there are no elements of degree three in $A^0_{PGL_2}(\Delta_{2,8})$ by (6.3), this is equivalent to asking that $\partial(t_8 x_4)$ is zero in

$A^1_{PGL_2}((P^6 \smallsetminus \Delta_{1,6}) \times P^1)$. As the boundary of x_4 is the unique element of degree 3 in $A^1_{PGL_2}((P^6 \smallsetminus \Delta_{1,6}) \times P^1)$ we can proceed with the usual induction, on $(P^i \smallsetminus \Delta_{1,i}) \times P^1$. The P^1 factor kills all elements of positive degree by (6.2), allowing us to conclude as in 2.1. $\square$

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