Brauer groups of moduli of hyperellictic curves, via cohomological invariants

Roberto Pirisi

KTH Royal Institute of Technology

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Notation

Unless otherwise stated, every scheme and stack is of finite type over a base field k. We denote $c = \mathrm{char}(k)$.

By ℓ we will always mean a positive integer not divisible by c.

If A is an abelian torsion group:

- A_ℓ is the subgroup of ℓ-torsion.
- ${}^cA$ is the subgroup of elements whose order is not divisible by c.

Unless otherwise stated, by $\mathrm{H}^i(X, F)$ we mean étale cohomology, or lisse-étale for Artin stacks.

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We say that two Azumaya algebras $\mathcal{E}, \mathcal{E}'$ are equivalent if $\mathcal{E} \otimes_X \mathrm{End}(\mathcal{F}) \simeq \mathcal{E}' \otimes_X \mathrm{End}(\mathcal{F}')$ for some free sheaves $\mathcal{F}, \mathcal{F}'$.

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The set of (equivalence classes of) Azumaya algebras forms a group with respect to tensor product. The identity is the class of M_n for any n and the inverse to $\mathcal{E}$ is the dual $\mathcal{E}^\vee$. We call it the *Brauer group* of X, denoted $\mathrm{Br}(X)$.

Azumaya algebras of rank n^2 are in a natural bijection with PGL_n-torsors. A trivial Azumaya algebra comes from a GL_n-torsor. Consider the exact sequence

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$$\mathrm{H}^1(X, \mathrm{GL}_n) \to \mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbb{G}_m)$$

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which shows that the Brauer group of X injects into $\mathrm{H}^2(X, \mathbb{G}_m)$.
Moreover, we know that for a Noetherian scheme the Brauer group is always torsion. Motivated by this, Grothendieck defined the *cohomological Brauer group* $\mathrm{Br}'(X)$ as the torsion subgroup of $\mathrm{H}^2(X, \mathbb{G}_m)$.

An important question is in which cases the inclusion $\mathrm{Br}(X) \subseteq \mathrm{Br}'(X)$ is an equality. There are some known counterexamples, among which:

- A proper three dimensional algebraic space over an algebraically closed field (Mathur).
- A normal but non-separated surface over the complex numbers (Edidin-Hasset-Kresch-Vistoli).

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These counterexamples are (almost) "minimal". Surjectivity is known for a large class of schemes:

Theorem (Gabber, de Jong)

Let X be a quasi-compact and separated scheme equipped with an ample invertible sheaf. Then

$$\mathrm{Br}(X) = \mathrm{Br}'(X).$$

The ℓ-torsion of $\mathrm{Br}'(X)$

If we restrict to the ℓ-torsion of $\mathrm{Br}'(X)$ we can use the étale exact sequence

$$1 \to \mu_\ell \to \mathbb{G}_m \xrightarrow{\cdot \ell} \mathbb{G}_m \to 1$$

to get the exact sequence

$$\mathrm{Pic}(X)/\ell \to \mathrm{H}^2(X, \mu_\ell) \to \mathrm{Br}'(X)_\ell \to 0.$$

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There is an important class of elements of $\mathrm{Br}'(X)$ which come from the Brauer group. Given a μ_ℓ-torsor $\alpha \in \mathrm{H}^1(X, \mu_\ell)$ and a $\mathbb{Z}/\ell\mathbb{Z}$ torsor $\beta \in \mathrm{H}^1(X, \mathbb{Z}/\ell\mathbb{Z})$, we can form the *cyclic algebra* $\mathcal{A}_{\alpha,\beta} \in \mathrm{Br}(X)$. The class of $\mathcal{A}_{\alpha,\beta}$ in $\mathrm{H}^2(X, \mu_\ell)$ is the cup product $\alpha \cdot \beta$.

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Regarding surjectivity of the Brauer map, we can put together results of Edidin-Hasset-Kresch-Vistoli and Kresch-Vistoli to obtain:

Theorem (EHKV, KV)

Let $\mathcal{X}$ be a smooth, separated, generically tame Deligne-Mumford stack with quasi-projective coarse moduli space. Then

$$^c\mathrm{Br}(\mathcal{X}) = {}^c\mathrm{Br}'(\mathcal{X}).$$

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This holds in particular for the stacks $\mathcal{M}_{1,1}$ and $\mathcal{H}_g$ in characteristic different from 2.

Theorem (DL–P)

Let r_g *be the remainder of* g *mod* 2*. Recall that the Picard group of* $\mathcal{H}_g$ *is cyclic of order* $2^{r_g}(4g+2)$*. We have:*

$$ {}^c\mathrm{Br}(\mathcal{H}_g) = {}^c\mathrm{Br}(k) \oplus {}^c\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{H}_g)) \oplus (\mathbb{Z}/2\mathbb{Z})^{1+r_g} $$

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Where

- The ${}^{c}\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{H}_g))$ component is given by cyclic algebras.
- The common $\mathbb{Z}/2\mathbb{Z}$ comes from BS_{2g+2}.
- The $\mathbb{Z}/2\mathbb{Z}$ appearing only for g odd comes from BPGL_2. It is the class of the universal conic over $\mathcal{H}_g$.

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Another way to see it is that α functorially assigns to each point $P \in \mathcal{M}(K)$ an element $\alpha(P) \in H^\bullet(K, D)$. The continuity condition requires α to “respect specialization”.

The set $\mathrm{Inv}^\bullet(\mathcal{X}, D)$ of cohomological invariants with coefficients in D inherits a group structure from $H^\bullet(-, D)$.
If $D = \mathbb{Z}/\ell\mathbb{Z}$, the cup product

$$\mathrm{H}^i(-, \mathbb{Z}/\ell\mathbb{Z}(i)) \otimes \mathrm{H}^j(-, \mathbb{Z}/\ell\mathbb{Z}(j)) \dot{\to} \mathrm{H}^{i+j}(-, \mathbb{Z}/\ell\mathbb{Z}(i+j))$$

endows the group of cohomological invariants with the structure of a graded-commutative ring.
In general, $\mathrm{Inv}^\bullet(\mathcal{X}, D)$ is a $\mathrm{Inv}^\bullet(\mathcal{X}, \mathbb{Z}/\ell\mathbb{Z})$-module.

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endows the group of cohomological invariants with the structure of a graded-commutative ring.
In general, $\mathrm{Inv}^\bullet(\mathcal{X}, D)$ is a $\mathrm{Inv}^\bullet(\mathcal{X}, \mathbb{Z}/\ell\mathbb{Z})$-module.
There is a map $H^\bullet(\mathcal{X}, D) \to \mathrm{Inv}^\bullet(\mathcal{X}, D)$ sending an element h to the cohomological invariant $\tilde{h}$ defined by $\tilde{h}(P) = h\mid_P \in H^\bullet(K, D)$.
In general this map is neither injective nor surjective.

For a map $\mathcal{X} \to \mathcal{Y}$, we have an obvious pullback of cohomological invariants given by composition.

Definition

A *smooth-Nisnevich* morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ is a smooth representable morphism of algebraic stacks such that for every point $\mathrm{Spec}(K) \to \mathcal{Y}$ there is a lifting $\mathrm{Spec}(K) \to \mathcal{X} \to \mathcal{Y}$.
The smooth-Nisnevich site of $\mathcal{X}$ is the site where the objects are smooth representable morphisms to $\mathcal{X}$ and the coverings are smooth-Nisnevich morphisms.

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Theorem (–)

The functor $\mathrm{Inv}^{\bullet}(-, D)$ *is a smooth-Nisnevich sheaf.*

Cohomological invariants as unramified cohomology

Let X be a smooth scheme over k. The *unramified cohomology* of X with coefficients in D is defined as:

$$\mathrm{H}^{\bullet}_{\mathrm{nr}}(X, D) = \mathrm{Ker} \bigoplus_{x \in X^{(1)}} \partial_x : H^{\bullet}(k(X), D) \to H^{\bullet}(k(X), D)$$

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Theorem (–)

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Let $\mathcal{X}$ be a smooth algebraic stack over k. Then $\mathrm{Inv}^{\bullet}(-, D)$ is the sheafification of $H^{\bullet}(-, D)$ on the smooth-Nisnevich site of $\mathcal{X}$.

Corollary

Let $f : \mathcal{Y} \to \mathcal{X}$ be a map of smooth algebraic stacks. If one of the following holds:

- *f is an affine bundle.*
- *f is an open immersion whose complement has codimension at least 2.*

We have $\mathrm{Inv}^{\bullet}(\mathcal{X}, D) = \mathrm{Inv}^{\bullet}(\mathcal{Y}, D)$.

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Proposition

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We have $\mathrm{Br}'(\mathcal{X})_\ell = \mathrm{Br}'(\mathcal{Y})_\ell$.

Cohomological Brauer as a sheaf

Let X be a smooth scheme over k. Consider the morphism of sites $(i_*, i^*) : X_{\text{ét}} \to X_{\mathrm{Zar}}$. It induces a Leray spectral sequence

$$\mathrm{H}^p_{\mathrm{Zar}}(X, \mathrm{R}^q i_* \mathbb{G}_m) \Rightarrow \mathrm{H}^{p+q}(X, \mathbb{G}_m).$$

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Now note that as X is regular we have $\mathrm{R}^1 i_* \mathbb{G}_m = 0$ and $\mathrm{H}^2_{\mathrm{Zar}}(X, \mathbb{G}_m) = 0$. This shows that

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Using the exact sequence

$$\mathrm{Pic}(X) \to \mathrm{H}^2(X, \mu_\ell) \to \mathrm{H}^2(X, \mathbb{G}_m)_\ell \to 0$$

we conclude that

$$\mathrm{H}^2_{\mathrm{nr}}(X, \mathbb{Z}/\ell\mathbb{Z}(-1)) \simeq \mathrm{H}^0_{\mathrm{Zar}}(X, \mathrm{R}^2 i_* \mu_\ell) \simeq \mathrm{H}^2(X, \mathbb{G}_m)_\ell.$$

Comparing Inv^2 and Br'

We just showed that for a smooth scheme X we have

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Let G be an affine algebraic group acting on a smooth scheme X, and V a representation of G, free on an open subset U whose complement has codimension $>> 0$. Then $X' = (X \times U)/G$ is a smooth scheme. We call it an *equivariant approximation* of $\mathcal{X} = [X/G]$. Note that:

- $\mathrm{Br}'(\mathcal{X})_\ell = \mathrm{Br}'(X')_\ell$.
- $\mathrm{Inv}^\bullet(\mathcal{X}, D) = \mathrm{Inv}^\bullet(X', D)$.

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Theorem (DL–P)

Let $\mathcal{X} = [X/G]$ be as above. Then

$$\mathrm{Br}'(\mathcal{X})_\ell = \mathrm{Inv}^2(\mathcal{X}, \mathbb{Z}/\ell\mathbb{Z}(-1)).$$

A family of hyperelliptic curves $C \to S$ comes equipped with an hyperelliptic involution $\iota : C \to C$. The quotient $C' = C/\iota$ is a family of smooth conics over S.
The critical locus W_C of the map $C \to C'$ is finite and étale over S, of degree $2g + 2$. We call it the Weierstrass divisor of C.

Quotient conics, Weierstrass divisors

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The functor sending C to C' induces a map from $\mathcal{H}_g$ to the stack BPGL_2, which classifies smooth families of conics. The map is trivial if g is even and smooth-Nisnevich if g is odd.

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The functor sending C to W_C induces a map from $\mathcal{H}_g$ to the stack BS_{2g+2}, which classifies finite étale schemes of degree $2g+2$. The map is always smooth-Nisnevich.

We want to understand the cohomological invariants of BS_{2g+2} and $BPGL_2$.
Consider the stack BO_n. The K-points of BO_n are in correspondence with couples (V, q) where V is an n-dimensional K-vector space and q a nondegenerate quadratic form.

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We may diagonalize q to obtain $q \sim (a_1, \dots, a_n)$. The coefficients $a_1, \dots, a_n \in K^*$.
Consider $a_1, \dots, a_n$ as elements of $K^*/(K^*)^2 = \mathrm{H}^1(K, \mu_2)$, and let $\alpha_i(V, q) = \lambda_i(a_1, \dots, a_n)$, where λ_i is the i-th standard symmetric function.

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Theorem (Garibaldi, Merkurjev, Serre)

$$\mathrm{Inv}^\bullet(\mathrm{BO}_n, D) = H^\bullet(k, D) \oplus \alpha_1 \cdot H^\bullet(k, D)_2 \oplus \ldots \oplus \alpha_n H^\bullet(k, D)_2.$$

Now consider the inclusion $S_{2g+2} \subset O_{2g+2}$ given by permutation matrices. It induces a map $BS_{2g+2} \to BO_{2g+2}$.

With a slight abuse of notation, denote by α_i both the cohomological invariant of BO_{2g+2} and its pullback to BS_{2g+2}.

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Note that the restrictions of α_{g+2} is a multiple of α_{g+1} and the restrictions of $\alpha_{g+3}, \ldots, \alpha_{2g+2}$ are zero.

In particular,

$$^c\mathrm{Br}(\mathrm{BS}_{2g+2}) = {}^c\mathrm{Br}(k) \oplus \mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$$

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In particular,

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. The invariants of BPGL_2 are simple: using $\mathrm{PGL}_2 \simeq \mathrm{SO}_3$ and $\mathrm{O}_3 \simeq \mathrm{SO}_3 \times \mu_2$ we get

Corollary (Garibaldi, Merkurjev, Serre)

$$\mathrm{Inv}(\mathrm{BPGL}_2, D) = H^\bullet(k, D) \oplus w_2 \cdot H^\bullet(k, D)_2.$$

In particular, $^c\mathrm{Br} = {}^c\mathrm{Br}(k) \oplus \mathbb{Z}/2\mathbb{Z}$. The nontrivial class in the Brauer group is the class of the universal conic

$$\left[\mathbb{P}^1/\mathrm{PGL}_2\right] \to \mathrm{BPGL}_2.$$

Theorem (P. '17, P.'18, Di Lorenzo '19, Di Lorenzo, P. '20)

Let k be a field of characteristic different from 2. The elements $1, \alpha_1, \ldots, \alpha_{g+1}$ generate a sub-module

$$I_g = H^\bullet(k, D) \oplus \alpha_1 \cdot H^\bullet(k, D)_{N_g} \oplus \bigoplus_{i=2}^{g+1} \alpha_i \cdot H^\bullet(k, D)_2$$

of $\mathrm{Inv}^\bullet(\mathcal{H}_g, D)$, where N_g is equal to $4g+2$ or $8g+4$ depending on g being respectively even or odd. For even g, we have

$$\mathrm{Inv}^\bullet(\mathcal{H}_g, D) = I_g \oplus \beta_{g+2} \cdot H^\bullet(k, D)_2$$

For odd g

$$0 \to I_g \oplus w_2 \cdot H^\bullet(k, D)_2 \to \mathrm{Inv}(\mathcal{H}_g, D) \to H^\bullet(k, D)_2$$

Where the last map lowers degree by $g+2$.

We are ready to describe the Brauer group of $\mathcal{H}_g$. Recall that the Brauer group of $\mathcal{H}_g$ is cyclic of order $4g+2$ if g is even and $8g+4$ if g is odd. Let r_g be the remainder of g mod 2.

Theorem (DL–P)

$$^c\mathrm{Br}(\mathcal{H}_g) = {}^c\mathrm{Br}(k) \oplus {}^c\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{H}_g)) \oplus (\mathbb{Z}/2\mathbb{Z})^{1+r_g}$$

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Where

- The ${}^c\mathrm{H}^1(k, \mathrm{Pic}(\mathcal{H}_g))$ component is given by cyclic algebras.
- The common $\mathbb{Z}/2\mathbb{Z}$ comes from BS_{2g+2}.
- The $\mathbb{Z}/2\mathbb{Z}$ appearing only for g odd comes from BPGL_2. It is the class of the universal conic over $\mathcal{H}_g$.

X equidimensional scheme of dimension d. Define

$$C^i(X, D) = \bigoplus_{x \in X^{(i)}} H^\bullet(k(x), D)$$

We have a complex

$$0 \to C^0(X, D) \xrightarrow{\partial} C^1(X, D) \xrightarrow{\partial} \dots \xrightarrow{\partial} C^d(X, D) \to 0.$$

We define the i-th codimensional Chow group with coefficients $A^i(X, D)$ as the i-th cohomology of the complex above. By definition

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All properties of ordinary Chow groups hold. If $D = \mathbb{Z}/\ell\mathbb{Z}$ and X is smooth they form a ring, otherwise they are an $A^\bullet(X,\mathbb{Z}/\ell\mathbb{Z})$-module. Morover, for a closed immersion i of pure codimension s:

$$\dots A^i(X,D) \xrightarrow{j^*} A^i(U,D) \xrightarrow{\partial} A^{i-s}(V) \xrightarrow{i_*} A^{i+1}(X) \dots$$

Set $V = \{27x^2 + 4y^3 = 0\} \subset \mathbb{A}^2$. The canonical Weierstrass form gives us a presentation $(\mathrm{char}(k) \neq 2, 3)$:

$$\mathcal{M}_{1,1} = \left[\mathbb{A}^2 \smallsetminus V/\mathbb{G}_m\right]$$

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Our plan to compute $\mathrm{Inv}^{\bullet}(\mathcal{M}_{1,1}, D)$ is computing the invariants of $\mathbb{A}^2 \setminus V$ and then imposing the gluing conditions.

Lemma

Assume $f : X \to Y$ is a universal homeomorphism. Then

$$f_* : A^\bullet(X, D) \to A^\bullet(Y, D)$$

is an isomorphism.

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In particular, $A^\bullet(V, D) = A^\bullet(\mathbb{A}^1, D)$.
Chow groups with coefficients,like ordinary Chow groups, are homotopy invariant, so

$$A^\bullet(\mathbb{A}^2, D) = A^\bullet(\mathbb{A}^1, D) = A^\bullet(\mathrm{Spec}(k), D).$$

Thus we have an exact sequence

$$0 \to \mathrm{H}^\bullet(k, D) \xrightarrow{j^*} A^0(\mathbb{A}^2 \setminus V, D) \xrightarrow{\partial} \mathrm{H}^\bullet(k, D) \to 0$$

Recall that by the Kummer sequence we have $\mathrm{H}^1(K,\mu_\ell) = K^*/(K^*)^\ell$. In particular we have an element $\gamma \in \mathrm{H}^1(k(x,y),\mathbb{Z}/\ell\mathbb{Z})$ given by the class $\{27x^2+4y^2\}$.
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It's easy to see that γ is unramified on $\mathbb{A}^2 \setminus V$ and $\partial\gamma = 1 \in A^0(V, \mathbb{Z}/\ell\mathbb{Z})$.
This gives us a splitting

$$\mathrm{H}^\bullet(k, D) \xrightarrow{j^*} A^0(\mathbb{A}^2 \setminus V, D) = \mathrm{H}^\bullet(k, D) \oplus \gamma \cdot \mathrm{H}^\bullet(k, D) \xrightarrow{\partial} \mathrm{H}^\bullet(k, D).$$

Showing that the cohomological invariants of $\mathbb{A}^2 \setminus V$ are generated by the trivial ones coming from the base and the product of α with the trivial invariants.

All that is left is to check the gluing conditions. Note that

$$\mathbb{A}^2 \setminus V \times_{\mathcal{M}_{1,1}} \mathbb{A}^2 \setminus V = \mathbb{A}^2 \setminus V \times \mathbb{G}_m$$

and the two projections are respectively the ordinary projection Pr_1 and the multiplication map m. The gluing conditions read

$$\mathrm{Pr}_1^* \alpha = m^* \alpha \in \mathrm{Inv}^\bullet(\mathbb{A}^2 \setminus V \times \mathbb{G}_m, D).$$

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Note that $m^*\gamma = \{27x^2t^{12} + 4y^2t^{12}\} = \gamma + 12\{t\}$. We have

$$\mathrm{Inv}^\bullet(\mathbb{A}^2 \setminus V \times \mathbb{G}_m, D) = \mathrm{Inv}^\bullet(\mathbb{A}^2 \setminus V, D) \oplus t \cdot \mathrm{Inv}^\bullet(\mathbb{A}^2 \setminus V, D)$$

So an element $x_0 + x_1\gamma$ glues $\Leftrightarrow 12x_1 = 0$.

We have just proven that

$$\operatorname{Inv}^{\bullet}(\mathcal{M}_{1,1}, D) = \mathrm{H}^{\bullet}(k, D) \oplus \{27x^2 + 4y^3\} \cdot \mathrm{H}^{\bullet}(k, D)_{12}.$$

Restricting to $D = \mathbb{Z}/\ell\mathbb{Z}(-1)$ and degree two, we retrieve the following result:

Theorem (Antieau, Meier 2016/18)

Suppose the characteristic of k is not 2 or 3. Then:

$${}^{c}\mathrm{Br}(\mathcal{M}_{1,1}) = {}^{c}\mathrm{Br}(k) \oplus {}^{c}\mathrm{H}^{1}(k, \mathbb{Z}/12\mathbb{Z}).$$

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However, our techniques are much simpler than theirs.

Thank you!