Roberto Pirisi - Research statement

Introduction

In the classical setting, algebraic geometry tries to classify (projective) algebraic varieties, i.e. subsets $X \subset \mathbb{P}^n(K)$ described by algebraic equations, where K is an algebraically closed field. A crucial objective is, given a specific kind of algebraic varieties, to parametrize their isomorphism classes in some geometrically meaningful way. This is often referred to as a *moduli problem*. Possibly the most famous moduli problem is to parametrize families of smooth curves, which led to the construction of the moduli space M_g of smooth curves.

A powerful breakthrough in moduli theory was the introduction by Deligne and Mumford [9] of the *moduli stack* $\mathscr{M}_g$ of smooth algebraic curves of genus g, which allowed for a relatively easy proof of their famous result that the variety M_g is irreducible. It soon became evident that the stack $\mathscr{M}_g$ was actually a much better object to work with than M_g, giving birth to the widespread use of *algebraic stacks*.

An algebraic stack is a generalization of a variety, where the points can have "intrinsic automorphisms". Using them instead of ordinary varieties to represent moduli problems allows to describe the problem much more faithfully, and to have good properties such as smoothness if the objects being described are well-behaved. For example, the moduli stack $\mathscr{M}_g$ is smooth, while the moduli space M_g is not. Algebraic stacks produce an algebro-geometric version of orbifolds, and also of the topological classifying spaces BG.

The price to pay for these advantages is an increased level of technicality, which requires mastery of both geometric and categorical arguments. Many areas have benefited from the theory of algebraic stacks, from the study of curves to abelian varieties to surfaces. Recently they are becoming of interest to physicists too, for example in string theory.

My work has concentrated on constructing and computing invariants for moduli stacks. Functorial invariants for moduli stacks are especially useful, as they will provide invariants for the families of objects being parametrized. Prime examples of invariants which are of interest are the Picard group of line bundles, the Chow groups (an algebraic version of singular homology), and étale cohomology with various kind of coefficients.

Past research

A major part of my research [11, 20–22] regards *Cohomological invariants*, a theory of arithmetic invariants which was classically associated with isomorphism classes of principal G-bundles over fields, and thus can be regarded as invariants of the moduli stack BG, which like its topological counterpart classifies principal G-bundles. I extended the theory to arbitrary algebraic stacks, and then computed the cohomological invariants of the stack of elliptic curves $\mathscr{M}_{1,1}$, of the stack of smooth genus two curves $\mathscr{M}_2$ and of the stacks $\mathscr{H}_g$ of smooth hyperelliptic curves of genus g for all even g and for $g = 3$. The computation for odd genus was completed by A. Di Lorenzo, and we recently gave a new explicit construction of the invariants for even genus which works over any field and allows us to compute the ring structure.

I also computed the Picard group of universal families of Abelian varieties and the Brauer group of the moduli stack of vector bundles over curves [14, 15] (joint with R. Fringuelli), studied the motivic classes of the classifying spaces of Spin_n and G_2-principal bundles [23] (joint with M. Talpo), and produced a "non-commutative" reconstruction theorem in the birational setting [8] (joint with J. Calabrese).

Present and future research

Computing the cohomological invariants of $\mathscr{M}_{\mathbf{g}}$:

A cohomological invariant σ of a moduli stack $\mathscr{M}$ can be thought of as an arithmetic equivalent to a characteristic class; given a family of objects $X \xrightarrow{\pi} S$ parametrized by $\mathscr{M}$ it provides an element $\sigma(\pi) \in \mathcal{H}(S)$, living in the unramified cohomology of S. In the classical case when $\mathscr{M} = \mathrm{BG}$ they were studied by many authors in relation to rationality problems and essential dimension, see for example [16–19].

The natural next steps after my computations in [11, 20–22] would be to give an explicit construction of the cohomological invariants of $\mathscr{H}_g$ for all odd g, and to compute the invariants of the stack $\mathscr{M}_3$ of smooth genus three curves. I plan to attack these questions using the new presentations of these stacks developed by Andrea Di Lorenzo, a student of Vistoli, as part of his PhD thesis.

Project 1 (joint with A. Di Lorenzo)**.** Completely describe cohomological invariants of $\mathscr{H}_g$ for all odd g and of $\mathscr{M}_3$.

An important property of cohomological invariants is that by a slightly more general definition, the degree two part retrieves the cohomological Brauer group BR'. Using the new explicit construction of the cohomological invariants of $\mathscr{H}_g$, it is possible to compute its cohomological Brauer group. In an upcoming joint work with A. Di Lorenzo [12], we show that over any field of characteristic zero

$$\text{If } g \text{ is even} \quad \mathrm{Br}'(\mathscr{H}_g) \simeq \mathrm{Br}'(k) \oplus \mathrm{H}^1(k, \mathbb{Z}/(4g+2)\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

$$\text{If } g \text{ is odd} \quad \mathrm{Br}'(\mathscr{H}_g) \simeq \mathrm{Br}'(k) \oplus \mathrm{H}^1(k, \mathbb{Z}/(4g+2)\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In the future, we plan to extend the computation to arbitrary fields and use it to explicitly compute the (ordinary) Brauer group of $\mathscr{H}_g$ by constructing explicit Azumaya algebras.

Project 2 (joint with A. Di Lorenzo)**.** Describe the Brauer group of $\mathscr{H}_g$ over any field.

Studying the invariants of $\mathscr{M}_g$ for general g will require a different approach. Let T_g be the profinite completion of the g-th Teichmüller group. There is a map $\mathscr{M}_g \to \mathrm{BT}_g$, which is an isomorphism from the point of view of étale homotopy type. This in particular induces maps $\mathscr{M}_g \to \mathrm{BG}$ for all finite quotients G of T_g. A natural subring of $\mathrm{Inv}^\bullet(\mathscr{M}_g)$ to study is the ring generated by the restrictions of the cohomological invariants of all such groups to those of $\mathscr{M}_g$.

Project 3. Study the subring of $\mathrm{Inv}^\bullet(\mathscr{M}_g)$ generated by the cohomological invariants of finite quotients of the Teichmüller group.

An algebraic geometry approach to Casson-type invariants

An important invariant in the study of three dimensional manifolds is the Casson invariant, which has been proven to have deep connections to Gauge theory and topological quantum field theories. Roughly speaking, it counts the classes of representations of the fundamental group $\pi(X)$ into SU(2), with multiplicities coming from a *Heegaard decomposition* of X. In the early 2000s, Curtis [6, 7] showed that using intersection theory one can construct corresponding invariants that counts representations in SL_2 or PSL_2, and that seeing the invariant as an intersection product offered extra insight on it. More recently, Abouzaid and Manolescu produced a vast generalization of the SL_2 invariant [1], using techniques coming from derived and symplectic geometry, which they call *Full Casson invariant.*

Both the Casson–Curtis invariant and the Abouzaid–Manolescu invariant can be extended to representations into general reductive algebraic groups if one accepts doing intersection theory (or symplectic geometry) on algebraic stacks rather than algebraic varieties. This will be the subject of an upcoming paper [2]. Studying these generalized invariants is the focus of a joint project with Paolo Aceto.

Project 4 (Joint with P. Aceto)**.** Study the generalized Casson and Full Casson invariants.

A Bittner presentation of the Grothendieck ring of good moduli morphisms

The Grothendieck ring of algebraic varieties is an important object in algebraic geometry. Motivic invariants factor through it and many are defined using it, such as motivic integrals. A powerful tool in creating invariants on the Grothendick ring is the *Bittner presentation*, which describes the ring in terms of smooth proper varieties and blow-ups. Recently Bergh [5] constructed a Bittner presentation for the Grothendieck ring of Deligne–Mumford stacks using *stacky blow-ups*. The situation for general algebraic stacks is more complicated, but one can consider the smaller ring of algebraic stacks which admit a *good moduli space*, as defined by Alper [3], and require the operations to respect the good moduli morphism. On this new ring one can use techniques such as saturated blow-ups and destackification to try to obtain a Bittner-type presentation, which is the subject of a work in progress with D. Rydh.

Project 5 (Joint with D.Rydh)**.** Construct a Bittner-type presentation for the Grothendieck ring of good moduli morphisms.

Motivic classes of classifying stacks:

In the late 2000s Ekedahl defined a modified Grothendieck ring of algebraic stacks $K_0(\mathrm{Stk}/k)$, which is a localization of the Grothendieck ring of varieties. Many motivic invariants factor through it, making it an important object of study.

The "expected class formula" for the class of BG predicts that it should be $\{\mathrm{G}\}^{-1}$ when G is connected and 1 when G is finite. There are counterexamples for finite groups, and it is conjectured that the formula should not hold in general for connected groups either. This problem seems to be morally related to a major problem in group theory, Noether's problem for connected algebraic groups.

The class of BG has been computed for $\mathrm{PGL}_2, \mathrm{PGL}_3$ and SO_n [4,13,24]. In a joint paper with Mattia Talpo [23] we showed that the problem of whether BSpin_n satisfies the expected class formula boils down to the same problem for a certain finite subgroup $\Delta_n \subset \mathrm{Spin}_n$. We conjecture that BSpin_n should violate the formula for $n \geq 15$.

Project 6 (joint with M. Talpo)**.** Prove that BSpin_n fails to satisfy the expected class formula for some $n \geq 15$.

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