Research statement

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Introduction

My main area of research is Algebraic Geometry. More precisely, my areas of interest are moduli spaces (and moduli stacks) with particular regard to their arithmetic and cohomological properties, algebraic groups and their representations, cycle theories, motivic classes, essential dimension and abelian categories. The fundamental connecting tissue between these areas is the use of *algebraic stacks*.

To study a geometric object X one often needs to consider families $\mathscr{X} \to S$ varying "continuously" over a base S, in which X is embedded as the inverse image of a special point $s_0 \in S$. This idea led, among many advances, to the powerful tool of moduli spaces. These are spaces (i.e., topological, differential, algebraic varieties...) whose points correspond to isomorphism classes of a given type of object. A moduli space M is *fine* if families over S of the objects it parametrizes correspond $1:1$ to maps from S to M. The very first examples of moduli spaces are the Grassmannian varieties $\mathrm{Gr}(V, r)$, parametrizing r-dimensional subspaces of a vector space V.

Moduli spaces have proven to be important not only in geometry (both algebraic, analytic and differential), but also in physics, where they have prominent roles in string theory (where the moduli spaces involved are actually algebro-geometric objects) and quantum field theory.

Since the late 1950s it became apparent that moduli spaces were insufficient for studying algebro-geometric objects with non-trivial automorphism, and in fact the presence of automorphisms is a fundamental obstruction to the existence of a fine moduli space. Moreover, if one constructs a (coarse) moduli space anyway, it will often be singular even if the objects being parametrized are smooth and nicely behaved.

These problems were solved thanks to the introduction of algebraic stacks. One can think of an algebraic stack as an algebraic variety where the points have intrinsic stabilizer groups, corresponding to the automorphisms of the objects being parametrized. Algebraic stacks produce an algebro-geometric version of orbifolds, and also of classifying spaces BG in topology. Moduli *stacks* regain the "fine" property, as well as smoothness. A prominent example is the moduli stack $\mathscr{M}_g$ of smooth algebraic curves of genus g, constructed by Deligne and Mumford [10], which is smooth and comes with a universal family $\mathscr{C}_g \to \mathscr{M}_g$ such that given a family of curves $C \to S$ there is a unique map $f : S \to \mathscr{M}_g$ with $f^*\mathscr{C}_g = C$. For comparison, the (coarse) moduli space M_g is not smooth and does not admit a universal family $C_g \to M_g$. These examples and many others have made algebraic stacks a central object of study.

Past research

My work has concentrated on constructing and computing invariants for algebraic stacks. Functorial invariants for moduli stacks are especially useful, as they will automatically provide invariants for the families of objects being parametrized through pullback. Prime examples of invariants which are of interest are the Picard group of line bundles, the Chow groups (an algebraic version of singular homology), and étale cohomology with various kind of coefficients.

A major part of my research [20–22] regards *Cohomological invariants*, a theory of arithmetic invariants which was classically associated with algebraic groups. The starting point of my investigation was to note that the cohomological invariants $\mathrm{Inv}^\bullet(G)$ of an algebraic group G should really be regarded as invariants of the corresponding algebraic stack BG, which like its topological counterpart classifies principal G-bundles. Starting from this observation it was natural to extend the theory to any algebraic stack. I then computed the cohomological invariants of the stack of elliptic curves $\mathscr{M}_{1,1}$, of the stack of smooth genus two curves $\mathscr{M}_2$ and of the stacks $\mathscr{H}_g$ of smooth hyperelliptic curves of genus g for all even g and for $g = 3$ when the base field is algebraically closed. Another student of Vistoli, A. Di Lorenzo, completed the computation for odd genus using a new presentation of $\mathscr{H}_g$, and recently we gave a new explicit description of the cohomological invariants of $\mathscr{H}_g$ for g even, which extends the result to arbitrary base fields and allows us to compute the multiplicative structure, which was previously unknown.

I also studied the Picard groups of universal families of Abelian varieties and the Brauer group of the moduli stacks of vector bundles over curves [15, 16] with R. Fringuelli, and the motivic classes (that is, the classes in an appropriate Grothendieck ring of isomorphism classes of algebraic stacks $K_0(\mathrm{Stk}/k)$) of the classifying spaces BSpin_n of Spin_n-principal bundles [23] (joint with M. Talpo). Finally, in a recent work with J. Calabrese [9] we prove that certain quotients of the abelian categories of coherent sheaves on a scheme retrieve its birational geometry.

Cohomological invariants

Consider an algebraic group G, and let Tors_G be the functor $\mathrm{Tors}_\mathrm{G} : (\mathrm{Fields}/k) \to (\mathrm{Sets})$ sending a field K/k to the set of isomorphism classes of principal G-bundles (more commonly called *torsors* in algebraic geometry) over K. In the modern definition [17], coined by Serre and Rost, a *cohomological invariant* $\alpha \in \mathrm{Inv}^\bullet(\mathrm{G})$ is a natural transformation between Tors_G and the Galois cohomology (with coefficients in μ_n) functor $\mathrm{H}^\bullet : (\mathrm{Fields}/k) \to (\mathrm{Sets})$.

Cohomological invariants can be thought of as an arithmetic equivalent to characteristic classes. They were studied in relation to both rationality problems and essential dimension by Serre, Rost, Merkurjev, Garibaldi, Totaro and many others.

It is natural to view the cohomological invariants of G as invariants of the classifying stack BG, as by definition the functor of isomorphism classes of G-torsors is the functor of (isomorphism classes of) points of BG.

I extended the definition to any algebraic stack [20, Def. 1.1] by defining a cohomological invariant of an algebraic stack $\mathscr{M}$ as a natural transformation from the functor of points $\mathrm{Pts}_{\mathscr{M}} : (\mathrm{Fields}/k) \to (\mathrm{Sets})$ to $\mathrm{H}^\bullet$ satisfying a natural continuity condition. This recovers the classical definition when $\mathscr{M} = \mathrm{BG}$.

When the stack $\mathscr{M}$ is smooth, I proved [20, Thm. 4.4] that the ring of cohomological invariants is equal to the sheafification of étale cohomology in an appropriate Grothendieck topology. This makes the ring of cohomological invariants a natural extension of unramified cohomology to algebraic stacks. Moreover, when $\mathscr{M}$ is a smooth quotient stack I proved [21, Prop. 2.10] that the ring of cohomological invariants is equal to the zero-dimensional part of the G-equivariant Chow ring with coefficients [18,24] in $\mathrm{H}^\bullet$, making it the main tool for computing cohomological invariants.

Computing the cohomological invariants of $\mathscr{M}_{\mathbf{g}}$

Using techniques coming from the study of equivariant Chow rings [14,19] and presentations produced by Arsie and Vistoli [4], and the theory of equivariant Chow rings with coefficients, I computed the cohomological invariants of the stack $\mathscr{M}_{1,1}$ of elliptic curves [20, Thm. 5.1], the stacks $\mathscr{H}_g$ of hyperelliptic curves of genus g when g is even [21, Thm. 4.1] and of the stack $\mathscr{H}_3$ of hyperelliptic curves of genus three [22, Thm. 3.12]. A. Di Lorenzo, completed the computation for odd genus using a new presentation of $\mathscr{H}_g$ [11], and recently we gave an explicit description of the cohomological invariants of $\mathscr{H}_g$ for g even [12], extending the result to arbitrary base fields and computing the multiplicative structure. The natural next steps would be to find an explicit description of the cohomological invariants of $\mathscr{H}_g$ for odd g, and to compute the invariants of the stack $\mathscr{M}_3$ of smooth genus three curves (note that $\mathscr{H}_2 = \mathscr{M}_2$). We plan to attack these questions using new presentations of these stacks being developed by Andrea Di Lorenzo as part of his Phd thesis.

Project 1 (joint with A. Di Lorenzo)**.** Explicitly describe the cohomological invariants of $\mathscr{H}_g$ for all odd g and compute the invariants of $\mathscr{M}_3$.

An important property of cohomological invariants is that by a slightly more general definition, the degree two part retrieves the cohomological Brauer group BR′. Using the new explicit construction of the cohomological invariants of $\mathscr{H}_g$, it is possible to compute its cohomological Brauer group. In an upcoming joint work with A. Di Lorenzo [?DilPir2], we show that over any field of characteristic zero

$$\text{If } g \text{ is even} \quad \mathrm{Br}'(\mathscr{H}_g) \simeq \mathrm{Br}'(k) \oplus \mathrm{H}^1(k, \mathbb{Z}/(4g+2)\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

$$\text{If } g \text{ is odd} \quad \mathrm{Br}'(\mathscr{H}_g) \simeq \mathrm{Br}'(k) \oplus \mathrm{H}^1(k, \mathbb{Z}/(4g+2)\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

In the future, we plan to extend the computation to arbitrary fields and use it to explicitly compute the (ordinary) Brauer group of $\mathscr{H}_g$ by constructing explicit Azumaya algebras.

Project 2 (joint with A. Di Lorenzo)**.** Describe the Brauer group of $\mathscr{H}_g$ over any field.

These techniques cannot be applied directly to $\mathscr{M}_g$ when g is high enough, as these stacks become of general type and this makes the existence of a "good" quotient stack presentation (i.e. one that makes for easy equivariant computations) for $\mathscr{M}_g$ unlikely. One way to get around this might be the following. Let T_g be the profinite completion of the g-th Teichmüller group. There is a map $\mathscr{M}_g \to \mathrm{BT}_g$, which is an isomorphism from the point of view of étale homotopy type. This in particular induces maps $\mathscr{M}_g \to \mathrm{BG}$ for all finite quotients G of T_g. A natural subring of $\mathrm{Inv}^\bullet(\mathscr{M}_g)$ to study is the ring generated by the restrictions of the cohomological invariants of all such groups to those of $\mathscr{M}_g$.

Project 3. Study the subring of $\mathrm{Inv}^\bullet(\mathscr{M}_g)$ generated by the cohomological invariants of finite quotients of the Teichmüller group.

An algebraic geometry approach to Casson-type invariants

An oriented three dimensional manifold Y always admits a *Heegaard decomposition* (Σ_g, U_1, U_2), where $U_1, U_2 \subset Y$ are two handlebodies intersecting on the smooth genus g surface Σ_g, and $U_1 \cup U_2 = Y$. The decomposition induces a pushout diagram of fundamental groups

$$\begin{array}{ccccc} & & F_g & & \\ & \nearrow & & \searrow & \\ \pi_1(\Sigma_g) & & & & \pi_1(Y) \\ & \searrow & & \nearrow & \\ & & F_g & & \end{array}$$

Let $\mathscr{X}_{\mathrm{irr}}(H)$ be the space of (irreducible) representations of H into a given group G. From the diagram above we obtain the cartesian diagram

$$\begin{array}{ccccc} & & \mathscr{X}_{\mathrm{irr}}(F_g) & & \\ & \nearrow & & \searrow & \\ \mathscr{X}_{\mathrm{irr}}(\pi_1(Y)) & & & & \mathscr{X}_{\mathrm{irr}}(\pi_1(\Sigma_g)) \\ & \searrow & & \nearrow & \\ & & \mathscr{X}_{\mathrm{irr}}(F_g) & & \end{array}$$

A well-known and important invariant of 3-manifolds, the Casson invariant, consists of intersecting the homology classes induced by these two subvarieties when $G = \mathrm{SU}_2$. Roughly speaking, it counts the classes of representations of the fundamental group $\pi(X)$ into SU(2), with multiplicities coming from the Heegaard decomposition of X. In the early 2000s, Curtis [7, 8] showed that using intersection theory one can construct the invariant when G is SL_2 or PSL_2, and that seeing the invariant as an algebraic intersection product offered extra insight on it. More recently, Abouzaid and Manolescu produced a vast generalization of the SL_2 invariant [1], using techniques coming from derived and symplectic geometry, which they call *Full Casson invariant*.

In an upcoming paper with Paolo Aceto [2], we will show that Both the Casson–Curtis invariant and the Abouzaid–Manolescu invariant can be extended to representations into general reductive algebraic groups if one accepts doing intersection theory (or symplectic geometry) on algebraic stacks rather than algebraic. Studying these generalized invariants is the focus of a joint project with Paolo Aceto.

Project 4 (Joint with P. Aceto)**.** Study the generalized Casson and Full Casson invariants.

Motivic classes of algebraic stacks

An important object of study in algebraic geometry is the Grothendieck ring of algebraic varieties over a field $K_0(\mathrm{Var}/k)$. Its elements are isomorphism classes of algebraic varieties, subject to three relations: a) if $U \subseteq X$ is an open immersion with closed complement V then $\{U\} + \{V\} = \{X\}$ b) we have $\{X \times_k Y\} = \{X\} \cdot \{Y\}$. Motivic invariants, such as the Euler characteristic, factor through it, so understanding it is an important objective in the study of algebraic varieties. One can enlarge it to contain more general objects, such as Deligne–Mumford stacks or Artin stacks. We are interested in these enlarged rings.

A Bittner presentation for the Grothendieck ring of good moduli morphisms

A powerful tool in creating invariants on the Grothendick ring is the *Bittner presentation*, which describes the ring in terms of smooth proper varieties and blow-ups. Namely, Bittner's theorem states that the classes of proper smooth varieties with the multiplicative relation and the *blow-up* relation

$$\{\widetilde{X}\} - \{E\} = \{X\} - \{Z\}$$

where E is the exceptional locus of the blow-up $\tilde{X}$ of X along Z, are sufficient to reconstruct the Grothendieck ring of varieties.

Recently Bergh [6] constructed a Bittner presentation for the Grothendieck ring of Deligne-Mumford stacks using *stacky blow-ups*, an extension of ordinary blow-ups which admit root stack constructions.

The situation for general algebraic stacks is more complicated, but one can consider the smaller ring of algebraic stacks $\mathscr{X}$ which admit a *good moduli space* *morphism* $\mathscr{X} \to X$, as defined by Alper [3], and consider the Grothendieck ring of morphisms $\mathscr{X} \to X$, where the operations are required to be compatible with the morphism. On this new ring one can use techniques such as saturated blow-ups and destackification to try to obtain a Bittner-type presentation, which is the subject of a work in progress with D. Rydh.

Project 5 (Joint with D.Rydh)**.** Construct a Bittner-type presentation for the Grothendieck ring of good moduli morphisms.

Classes of classifying stacks in Ekedahl's Grothendieck ring of algebraic stacks

In the late 2000s Ekedahl defined a modified Grothendieck ring of algebraic stacks $K_0(\mathrm{Stk}/k)$. Its elements are isomorphism classes of algebraic stacks, subject to three relations: a) if $\mathscr{U} \subseteq \mathscr{X}$ is an open immersion with closed complement $\mathscr{V}$ then $\{\mathscr{U}\} + \{\mathscr{V}\} = \{\mathscr{X}\}$ b) we have $\{\mathscr{X} \times_k \mathscr{Y}\} = \{\mathscr{X}\} \cdot \{\mathscr{Y}\}$ c) if $\mathscr{E} \to \mathscr{X}$ is a vector bundle of rank d then $\{\mathscr{E}\} = \{\mathbb{A}^d \times_k \mathscr{X}\}$ (note that this is always true for varieties). Due to the last relation Ekedahl's ring is a localization of the Grothendieck ring of algebraic varieties.

Computing the class of the classifying stacks BG in this ring is an open problem. There is an "expected class formula" saying that the class of BG should be $\{\mathrm{G}\}^{-1}$ when G is connected and 1 when G is finite. It holds when G is special, but there are counterexamples for finite groups, and the formula is expected to fail for connected groups too, even though no counterexample is known. This problem seems to be (at least morally) related to a major problem in group theory, Noether's problem for connected algebraic groups, which asks whether given a connected algebraic group G with a generically free representation V the quotient V/G is rational. A negative answer is expected for this question, but no example is known.

The class of BG has been computed for connected groups in the cases of PGL_2 and PGL_3 by Bergh [5], in the case of SO_n for odd n by Dhillon and Young [13] and in the case of SO_n for all n by Talpo and Vistoli [25]. In a joint paper with Mattia Talpo we computed [23, Thm. 3.1, 3.8] the classes of $\mathrm{BG}_2, \mathrm{BSpin}_7, \mathrm{BSpin}_8$, and showed [23, Thm. 4.5] that for any n the problem of whether BSpin_n satisfies the expected class formula boils down to the same problem for a certain finite subgroup $\Delta_n \subset \mathrm{Spin}_n$. We conjecture that BSpin_n should violate the formula for all $n \geq 15$.

Project 6 (joint with M.Talpo)**.** Prove that BSpin_n fails to satisfy the expected class formula for some $n \geq 15$.

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