# Review session Dec. 11

## 1) Triple integrals

#6 From Exam W 2006 T1

$$\int_0^1 \int_0^{1-x/2} \int_0^{4-2x-4z} f(x,y,z)\, dy\, dz\, dx$$

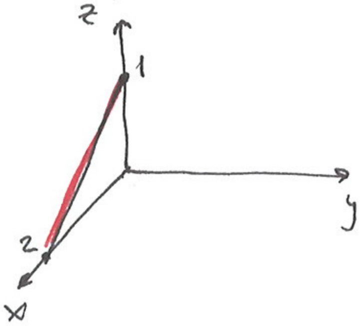
1) Sketch the domain
2) Rewrite as $\iiint f \, dz\, dx\, dy$

We have: $0 \le x \le 1$

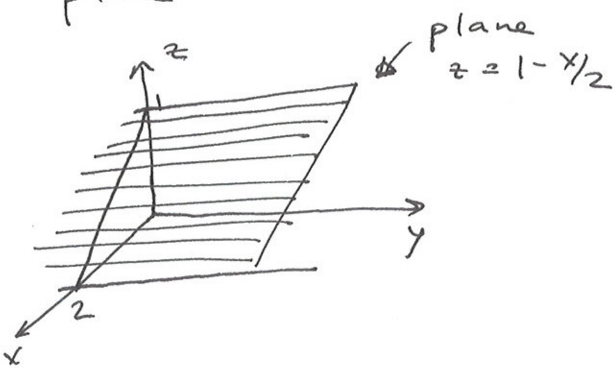
$0 \le z \le 1 - x/2$ plane parallel to y-axis

$0 \le y \le 4 - 2x - 4z$ plane

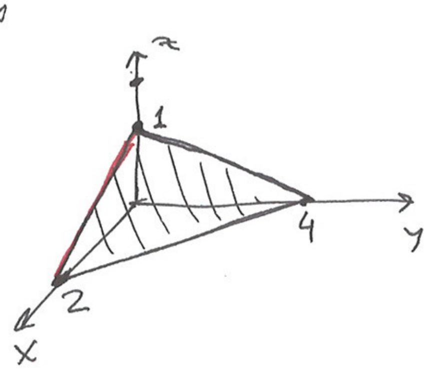
1)




2)




3)




sketch $y = 4 - 2x - 4z$

$y + 2x + 4z = 4$

find intercepts:

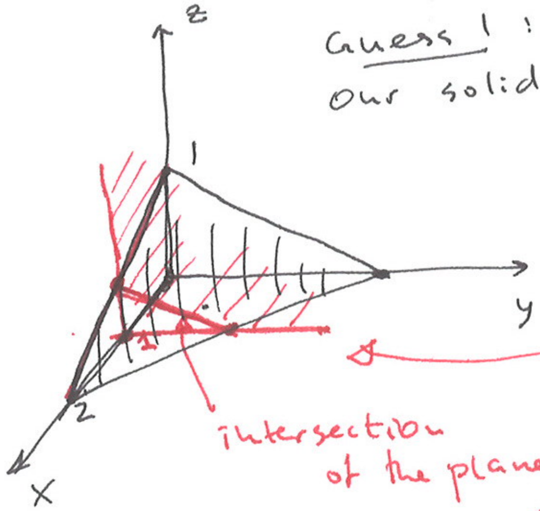
$y = 4$

$2x = 4, \; x = 2$

$4z = 4, \; z = 1.$

Note: the line in the xz-plane through $(2,0,0)$ and $(0,0,1)$ is common.

(the red line).



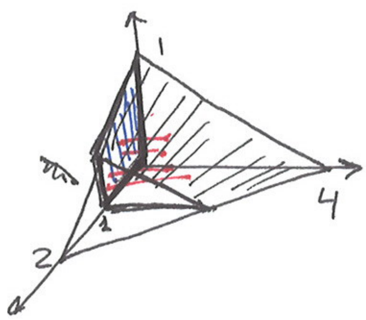

Guess 1:
Our solid is this tetrahedron in the first octant.

Correction! $0 \le x \le 1$!

$x=1$ is a plane parallel to y and z-axes.

intersection of the plane $x=1$ with $y+2x+4z=4$.

Guess 2:

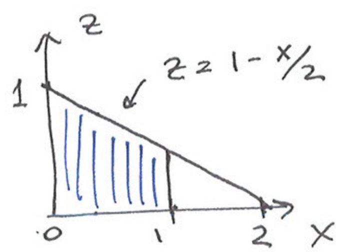



Compare to the original integral:

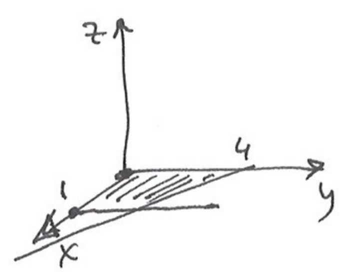
$$\underbrace{\int_0^1 \int_0^{1-x/2}}\int_0^{4-2x-4z} f_{..}\, dy\, dz\, dx$$

these limits describe the blue domain in the xz-plane

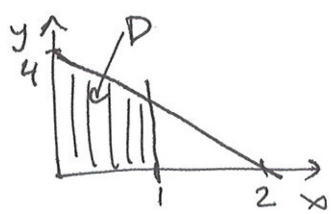
in the xz plane

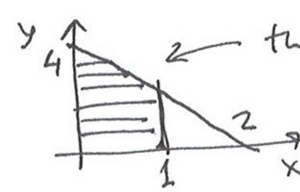



2) Rewrite as $\iiint_D \, dz\, dx\, dy$




What is the domain in the xy-plane we need to integrate over?
(projection onto the xy-plane).
Here it coincides with the cross-section with the xy-plane.

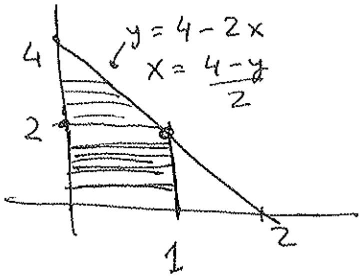



the order of integration is dx dy.

So, on the outside, get:

$$\iint_D \int \qquad dz \; dx \; dy$$




$$\boxed{\iint_D = \int_0^2 \int_0^1 \qquad dx\,dy + \int_2^4 \int_0^{\frac{4-y}{2}} \qquad dx\,dy}$$

Now, what about the inner integral?

given: $0 \le y \le 4 - 2x - 4z$

have to rewrite this constraint

$y = 4 - 2x - 4z$ so that

$z$ is expressed in terms of $x, y$.

$$z = \frac{4 - 2x - y}{4}.$$

Put it together:

$$\int_0^2 \int_0^1 \int_0^{\frac{4-2x-y}{4}} f(x,y,z)\, dz\, dx\, dy$$

$$+ \int_2^4 \int_0^{\frac{4-y}{2}} \int_0^{\frac{4-2x-y}{4}} f(x,y,z)\, dz\, dx\, dy.$$

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Advice: If you are just given a solid and asked to set up an integral, choose convenient order.

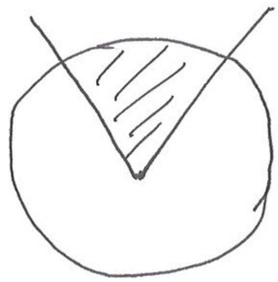
1) If given inequalities, put the variable that is the most used on the outside.

e.g. $y \le f(x,z)$, $z \le g(x)$. $\rightarrow$ put $x$ on the outside.

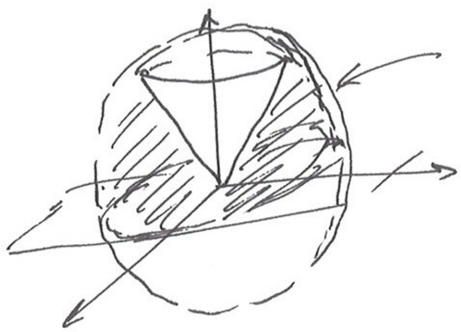
2) Draw the domain for the outside double integral. Choose convenient order.

Wording: "bounded below by" = "above"

"bounded above by" = "below".



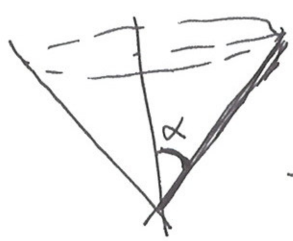
solid bounded below by the cone and bounded above by the sphere.



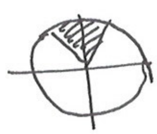
below the cone
above the xy-plane
"
bounded above by the cone,
bounded below by the xy-plane.

Set up both of these in spherical coord

Suppose the sphere has radius R. The cone has angle $\alpha$ at the vertex.







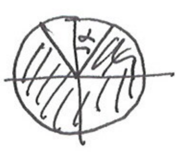
1) inside the sphere, above the cone:

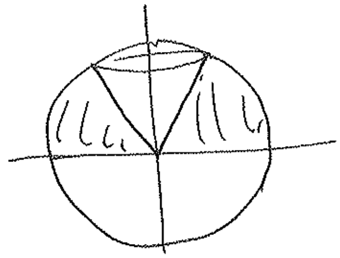
$$\int_0^{2\pi}\int_0^{R}\int_0^{\alpha} f \cdot \underbrace{\rho^2 \sin\varphi}\, d\varphi\, d\rho\, d\theta$$

from "dV" in spherical coords.

2) $$\int_0^{2\pi}\int_0^{R}\int_{\alpha}^{\pi} f(\rho,\varphi,\theta)\, \rho^2 \sin\varphi \, d\varphi\, d\rho\, d\theta$$

– below the cone, inside the sphere.



3) 

$$\int_0^{2\pi}\int_0^{R}\int_{\alpha}^{\pi/2} f \cdot \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta$$

(xy-plane: $\varphi = \frac{\pi}{2}$ in spherical).

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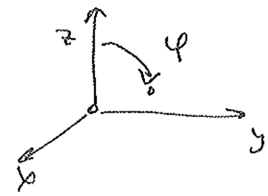
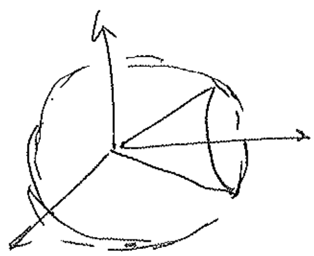
Question: When converting to spherical coords, traditionally, $z$ is "special":

$z = \rho\cos\varphi$
$x = \rho \sin\varphi\cos\theta$
$y = \rho\sin\varphi\sin\theta$




What if we have a cone around the $y$-axis?



could "rename $x, y, z$":

sometimes this gives a "$-$"

$\tilde{\varphi}, \tilde{\theta}$

$y = \rho\cos\tilde{\varphi}$
$z = \rho\sin\tilde{\varphi}\sin\tilde{\theta}$
$x = \rho\sin\tilde{\varphi}\cos\tilde{\theta}$.

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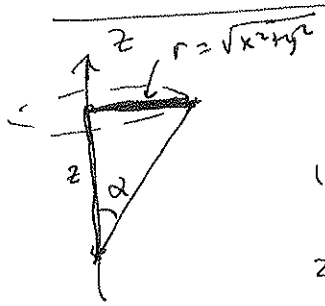
About cones and angles:




$z^2 = c(x^2+y^2)$

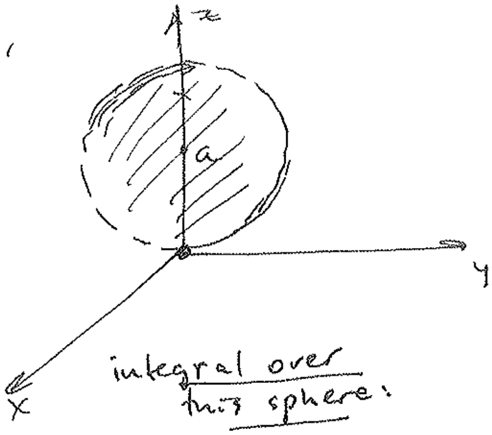
1. $\dfrac{\sqrt{x^2+y^2}}{z} = \tan(\alpha)$

2. $z = \sqrt{c(x^2+y^2)}$

$\dfrac{\sqrt{x^2+y^2}}{z} = \dfrac{1}{\sqrt{c}}$

so, $\boxed{\dfrac{1}{\sqrt{c}} = \tan(\alpha)}$

Also,




$$(z-a)^2 + x^2 + y^2 = a^2$$

$$z^2 - 2az + \cancel{a^2} + x^2 + y^2 = \cancel{a^2}$$

$$\underbrace{z^2 + x^2 + y^2}_{\rho^2} = \underbrace{2az}_{2a\rho\cos\varphi}$$

$$\boxed{\rho = 2a\cos\varphi}$$

integral over this sphere:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\varphi} \rho^2 \sin\varphi \, d\rho \, \underline{d\varphi} \, d\theta$$

important: have the integral $d\rho$ inside the integral $d\varphi$.

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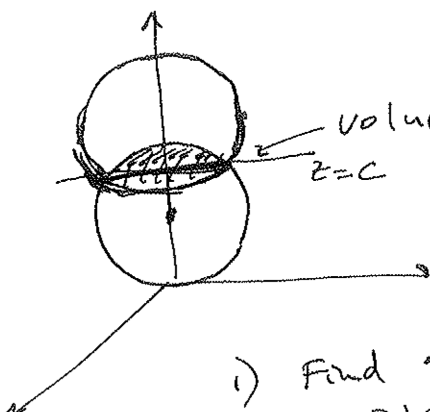
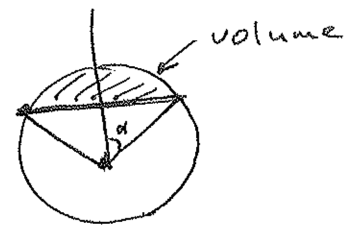
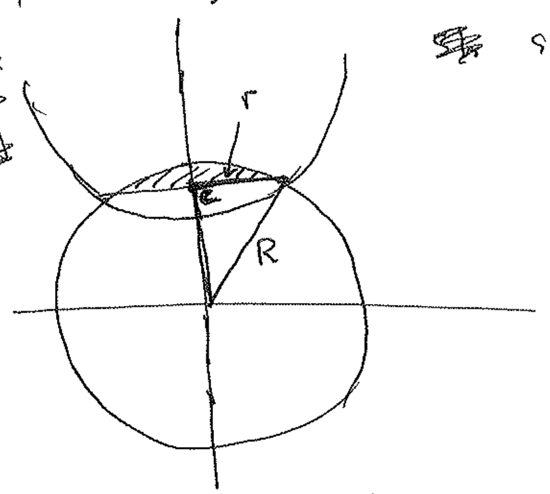
Question:




1) Find ~~z~~, the horizontal plane $z=c$ where the spheres intersect.

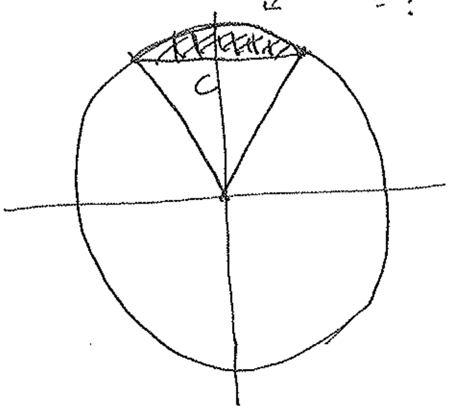
2) use cylindrical coords:



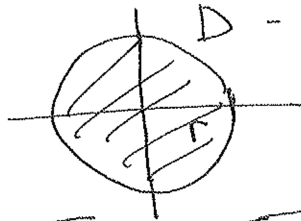

spheres of radius R intersect at "height" c

i) Find $r$ = radius of the circle of intersection.

$r = \sqrt{R^2 - c^2}$




$$\iint_D \int_c^{\sqrt{R^2-(x^2+y^2)}} 1 \; dz\, dx\, dy$$ ← the sphere




D - projection of our cap onto the xy-plane

$$= \int_0^{\sqrt{R^2-c^2}}\int_0^{2\pi}\int_c^{\sqrt{R^2-r^2}} 1\cdot r\, dz\, d\theta\, dr$$

cylindrical $(r, \theta, z)$

$$= \int_0^{\sqrt{R^2-c^2}} \int_0^{2\pi} \left(\sqrt{R^2-r^2} - c\right)\cdot r \; d\theta\, dr$$

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From Math 101: 1) substitution, e.g. $\boxed{\begin{matrix} x^2 = u \\ 2x\,dx = du \end{matrix}}$

2) useful $\int \sqrt{a^2-x^2}\, dx$

$\int \frac{1}{\sqrt{a^2-x^2}}\, dx$

3) $\int \cos\theta$, $\int \sin\theta$, $\int \cos^2\theta$, $\int \sin^2\theta$ ...

higher powers: will give hints.

4) Integration by parts.

Chain rule

April 2012 # 3

Given: $z = f(x, y)$ $\quad f_x(2,1) = 5$

$x = 2t^2$ $\quad f_y(2,1) = -2$ $\quad f_{yy}(2,1) = -4$

$y = t^3$ $\quad f_{xx}(2,1) = 2$

$f_{xy}(2,1) = 1$

Find: $\dfrac{d^2 z}{dt^2}$ at $t = 1$. (Note: z ends up being a function of single variable t).

" $z''(t)$ at $t=1$ " $= z''(1)$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Note: when $t = 1$, get $x = 2 \cdot 1^2 = 2$, $y = 1^3 = 1$, $(x,y) = (2,1)$.

$$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 3t^2$$

Plug this in:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot 4t + \frac{\partial f}{\partial y} \cdot 3t^2$$

(Note: too early to plug in values for $f_x$, $f_y$).

Differentiate

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) \cdot 4t + \frac{\partial f}{\partial x} \cdot 4 + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right) \cdot 3t^2 + \frac{\partial f}{\partial y} \cdot 6t$$

use product rule

$$= \left(\frac{\partial^2 f}{\partial x^2} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dt}\right) 4t + \frac{\partial f}{\partial x} \cdot 4$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dt}\right) \cdot 3t^2 + \frac{\partial f}{\partial y} \cdot 6t$$

$$= \left(f_{xx} \cdot 4t + f_{xy} \cdot 3t^2\right) \cdot 4t + 4 f_x$$

$$+ \left(f_{xy} \cdot 4t + f_{yy} \cdot 3t^2\right) \cdot 3t^2 + 6t\, f_y$$

Now: plug in $t = 1$ into all fns of $t$

$(x,y) = (2,1)$ into all functions of $(x,y)$.

Get: $$\left(f_{xx}(2,1) \cdot 4 + f_{xy}(2,1) \cdot 3\right) \cdot 4$$

$$+ 4 f_x(2,1) + \left(f_{xy}(2,1) \cdot 4 + f_{yy}(2,1) \cdot 3\right) \cdot 3$$

$$+ 6 f_y(2,1)$$

$$= (2 \cdot 4 + 1 \cdot 3) \cdot 4 + 4 \cdot 5 + \cdots$$

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Lagrange primitive question:

Find the distance from a point to a plane in space.

EX: Given $2z + x - y = 4$

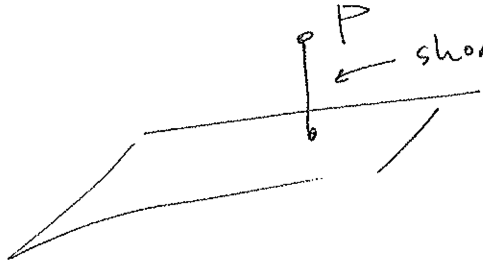
Find the distance from $(3,0,2)$ to this plane.

Use analysis, not geometry:

$$f(x,y,z) = (\text{dist from } (x,y,z) \text{ to } P)^2$$

$$= (x-3)^2 + (y-0)^2 + (z-2)^2$$

$$= (x-3)^2 + y^2 + (z-2)^2$$




shortest of all distances from P to points on the plane.

Need to find min $f$ subject to the constraint

$2z + x - y = 4$; $g(x,y,z) = 2z + x - y - 4$

$\bar{\nabla} f = \lambda \bar{\nabla} g$

$\bar{\nabla} f = \langle 2(x-3),\ 2y,\ 2(z-2) \rangle$

$\bar{\nabla} g = \langle 1, -1, 2 \rangle$

Get:

$$\begin{cases} 2(x-3) = \lambda \\ 2y = -\lambda \\ 2(z-2) = 2\lambda \\ 2z + x - y = 4 \end{cases}$$

Express everything through $\lambda$, plug in into the last equ.

$$\begin{cases} x = \frac{\lambda+6}{2} \\ y = -\lambda/2 \\ z = \frac{2\lambda+4}{2} = \lambda+2 \\ 2(\lambda+2) + \frac{\lambda+6}{2} + \frac{\lambda}{2} = 4 \end{cases}$$

$3\lambda + 7 = 4$

$\lambda = -\frac{3}{3} = -1.$

$x = 5/2$, $y = 1/2$, $z = 1$.

Answer: $\sqrt{(5/2 - 3)^2 + (1/2 - 0)^2 + (1-2)^2} = \dots$

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Math club: $\boxed{\begin{array}{lll} \text{Thurs} & 11-1 & \text{MATX 1119} \\ \text{Fri} & 11-1 & \end{array}}$