This midterm has ***5 questions*** *on* ***6 pages****, for a total of 50 points.*

Duration: 80 minutes

- Write your name on **every** page.
- **you need to show enough work to justify your answers.**
- Continue on the back of the previous page if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): ______________________________

Student-No: ______________________________

Signature: ______________________________

Section number: ______________________________

Name of the instructor: ______________________________

Question:	1	2	3	4	5	Total
Points:	12	12	9	7	10	50
Score:						

1. Let $F(x, y, z) = e^{3x}\sqrt{yz}$.

$\boxed{\text{3 marks}}$ (a) Find the linearization (linear approximation) of $F(x, y, z)$ near the point $P = (0, 25, 1)$.

$\boxed{\text{3 marks}}$ (b) Use the answer from (a) to estimate $F(0.01, 25.02, 0.99)$.

$\boxed{\text{2 marks}}$ (c) Write the equation of the level surface of F that contains the point P.

$\boxed{\text{4 marks}}$ (d) Find an equation of the tangent plane at P to the level surface of the function F that contains this point.

Solution: (a). We have $F_x = 3e^{3x}\sqrt{yz}$, $F_y = \frac{\sqrt{z}e^{3x}}{2\sqrt{y}}$, $F_z = \frac{\sqrt{y}e^{3x}}{2\sqrt{z}}$. Evaluating these partial derivatives at P, we get $F_x|_P = 15$, $F_y|_P = 0.1$, $F_z = 5/2 = 2.5$. We also have $F(0, 25, 1) = 5$. Then the linearization of F at P is

$$L(x, y, z) = 5 + 15x + 0.1(y - 25) + 2.5(z - 1).$$

(b). Plugging in the point $(0.01, 25.02, 0.99)$ into L, we get

$$F(0.01, 25.02, 0.99) \simeq 5 + 0.15 + 0.002 - 0.025 = 5.127.$$

(c). $F(x, y, z) = 5$.

(d). The gradient of F at P is normal to the tangent plane to the level surface; we have $\nabla_P F = \langle 15, 0.1, 2.5\rangle$. Then the equation of the tangent plane is:

$$15x + 0.1(y - 25) + 2.5(z - 1) = 0.$$

Note that this is also $L(x, y, z) = 5$.

2. A slug is crawling on the flat metal surface with the temperature of the surface at a point (x, y) given by $T = xe^{x^2+y}$.

3 marks (a) At the point $(1, 2)$, what is the direction of the greatest decrease of the temperature? (The answer should be a unit vector).

4 marks (b) Find the rate of change of the temperature the slug would experience if it crawled from the point $(1, 2)$ at speed 1 in the direction of the vector $\langle 3, 4\rangle$.

5 marks (c) Suppose at a time t (in seconds) the slug is at the point $(x(t), y(t))$ with $x(t) = 0.1t$, $y(t) = 0.02t^2$. What is the rate of change of temperature the slug experiences as it passes through the point $(1, 2)$?

Solution: We find the gradient of T at $(1, 2)$: $T_x = 2x^2e^{x^2+y}+e^{x^2+y} = e^{x^2+y}(2x^2+1)$, $T_y = xe^{x^2+y}$, and $\nabla_{(1,2)}T = \langle 3e^3, e^3\rangle$.

(a).The unit vector in the direction opposite to the gradient is $u = \langle -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\rangle$.

(b). The unit vector in the direction of $\langle 3, 4\rangle$ is $u_1 = \langle 3/5, 4/5\rangle$. Then

$$D_{u_1}T|_{(1,2)} = \nabla_{(1,2)}T \cdot u_1 = 9e^3/5 + 4e^3/5 = 13e^3/5.$$

Since the slug is crawling at speed 1, the rate of change of temperature it experiences is exactly the directional derivative in the direction it is crawling in, so the answer is $13e^3/5$.

(c). By Chain Rule,

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}.$$

Now, let us find the time t at which the slug is at $(1, 2)$. We have $0.1t = 1$ and $0.02t^2 = 2$, so $t = 10$. Then we have:

$$\frac{dT}{dt}|_{t=10} = \frac{\partial T}{\partial x}|_{(1,2)}\frac{dx}{dt}|_{t=10} + \frac{\partial T}{\partial y}|_{(1,2)}\frac{dy}{dt}|_{t=10}.$$

Compute $\frac{dx}{dt} = 0.1$, $\frac{dy}{dt} = 0.04t$. Finally, we get

$$\frac{dT}{dt}|_{t=10} = 3e^3 \cdot 0.1 + e^3 \cdot 0.04 \cdot 10 = 0.7e^3.$$

4 marks 3. (a) Let $w = ve^u$, and let $u(x,y,z) = x^3/z$, $v(x,y,z) = y^3/z$. Find $\frac{\partial w}{\partial x}$.

5 marks (b) Let $w = f(u,v)$, where f is a differentiable function, and $u(x,y,z) = x^3/z$, $v(x,y,z) = y^3/z$. Compute $(xw_x + yw_y)/w_z$.

Solution: (a). We have:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x};$$

$$\frac{\partial u}{\partial x} = \frac{3x^2}{z}; \quad \frac{\partial v}{\partial x} = 0; \quad \frac{\partial w}{\partial u} = ve^u; \quad \frac{\partial w}{\partial v} = e^u.$$

Then,

$$\frac{\partial w}{\partial x} = ve^u\frac{3x^2}{z} = \frac{3x^2y^3e^{x^3/z}}{z^2}.$$

(b).

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}; \quad \frac{\partial u}{\partial x} = \frac{3x^2}{z}; \quad \frac{\partial v}{\partial x} = 0;$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = 0; \quad \frac{\partial v}{\partial y} = \frac{3y^2}{z};$$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial z}; \quad \frac{\partial u}{\partial z} = -\frac{x^3}{z^2}; \quad \frac{\partial v}{\partial x} = -\frac{y^3}{z^2}.$$

Then:

$$xw_x + yw_y = \frac{\partial w}{\partial u}\frac{3x^3}{z} + \frac{\partial w}{\partial v}\frac{3y^3}{z}, \text{and}$$

$$\frac{xw_x + yw_y}{w_z} = -\frac{\frac{\partial w}{\partial u}\frac{3x^3}{z} + \frac{\partial w}{\partial v}\frac{3y^3}{z}}{\frac{\partial w}{\partial u}\frac{x^3}{z^2} + \frac{\partial w}{\partial v}\frac{y^3}{z^2}} = -3z.$$

$\boxed{\text{7 marks}}$ 4. Find and classify all the critical points of $f(x, y) = \cos x + y^2$.

Solution: $f_x = -\sin(x)$, $f_y = 2y$. Then the critical points are $(\pi k, 0)$, where k is an integer. $f_{xx} = -\cos(x)$, $f_{xy} = 0$, $f_{yy} = 2$. Then $D > 0$ if $-\cos(x) > 0$, that is, when k is odd; $D < 0$ if $-\cos(x) < 0$, that is, k is even. Then we have local minima at the points of the form $(\pi(2n + 1), 0)$, and saddle points at $(2\pi n, 0)$ for all integers n.

10 marks 5. Produce the complete list of points where the absolute max or min of $f(x, y) = x^2 + 3xy - 2y^2$ on the triangle T with vertices $(-1, 2)$, $(-1, -1)$ and $(2, -1)$ could occur; **do not evaluate the function at these points**.

Solution: $f_x = 2x+3y$, $f_y = 3x-4y$. The only critical point is $(0, 0)$. Now, analyze the boundary.

1. The bottom edge, $y = -1$, $-1 < x < 2$. Get $f(x, -1) = x^2 - 3x - 2$; this function has a critical point at $x = 3/2$. This point is within the interval $[-1, 2]$, so we add the point $(3/2, -1)$ to the list.

2. The left edge, $x = -1$, $-1 < y < 2$. We have $f(-1, y) = 1 - 3y - 2y^2$. The critical point of this function is at $y = -3/4$; we add the point $(-1, 3/4)$ to the list.

3. $y = 1 - x$, $-1 < x < 2$. Get

$$f(x, 1-x) = x^2+3x(1-x)-2(1-x)^2 = x^2+3x-3x^2-2-2x^2+4x = -4x^2+7x-2.$$

The critical point of this function is at $x = 7/8$, so we add the point $(7/8, 1/8)$ to the list.

4. Finally, all the vertices are on the list. Thus, the list of points where max/min could occur is $(0, 0); (3/2, -1), (-1, 3/4), (7/8, 1/8), (-1, 2), (-1, -1), (2, -1)$.