Math 263 Midterm I

Problem 1

(a) Find the parametric equations for the line of intersection of the two planes,

$$z = x + y, \quad 2x - y = 1$$

(b) Calculate the angle between the planes.
(c) Calculate the shortest distance between the line of intersection and the point $(0, 0, 0)$.

Solution:

(a) the two planes have normal vectors $< 1, 1, -1 >$ and $< 2, -1, 0 >$, respectively. A vector in the direction of the line is therefore

$$< 1, 1, -1 > \ \times \ < 2, -1, 0 > = < -1, -2, -3 >$$

If we put $x = 0$ in both equations for the planes, then we have $y = z = -1$, implying that $(0, -1, -1)$ lies on the line. Thus,

$$x = -t, \quad y = -1 - 2t, \quad z = -1 - 3t$$

(b) The angle between the plane is equivalent to the angle between the normal vectors, θ, which, using the dot product, is given by

$$\cos\theta \equiv \frac{< 1, 1, -1 > \ \cdot \ < 2, -1, 0 >}{\sqrt{(1+1+1)(4+1)}} = \frac{1}{\sqrt{15}}.$$

(c) The line has vector equation,

$$\mathbf{r} =< 0, -1, -1 > \ + t < -1, -2, -3 >$$

This is also the vector from the origin to the point closest to the origin for a certain value of t, in which case $\mathbf{r}$ should be orthogonal to $< -1, -2, -3 >$. Since the dot product of $\mathbf{r}$ and $< -1, -2, -3 >$ then vanishes, we find

$$t = -\frac{5}{14},$$

giving $\mathbf{r} =< 5, -4, 1 > /14$ and the distance $\sqrt{42}/14$.

Problem 2

(a) Show that the curves represented by $\mathbf{r}_1(t) =< t, t^2, t^2 >$ and $\mathbf{r}_2(t) =< \sin t, 1+\cos t, 1 >$ intersect at point P: $(1, 1, 1)$.

(b) Find the angle of intersection of the two curves at P.

(c) If $\mathbf{r}_1(t)$ represents the position of a particle moving in space, compute (i) its speed v at P; (ii) the tangential and normal components of its acceleration at P.

Solution:

(a) At $t = 1$, $\mathbf{r}_1(t) =< 1, 1, 1 >$. At $t = \pi/2$, $\mathbf{r}_2(t) =< 1, 1, 1 >$. Thus, the two curves do intersect at P, even though the two position vectors arrive at P at different values of their respective parameters. This tests the students' understanding of parametric equations for space curves.

(b) $\mathbf{r}_1' =< 1, 2t, 2t >=< 1, 2, 2 >$ at P. $\mathbf{r}_2' =< \cos t, -\sin t, 0 >=< 0, -1, 0 >$ at P. Their angle θ is computed from the "cosine rule" for dot product:

$$\cos\theta = \frac{\mathbf{r}_1' \cdot \mathbf{r}_2'}{|\mathbf{r}_1'||\mathbf{r}_2'|} = \frac{-2}{3 \times 1} = -\frac{2}{3}.$$

So $\theta = \arccos(-2/3)$.

(c) The velocity vector is the same as the tangent vector: $\mathbf{r}_1' =< 1, 2t, 2t >=< 1, 2, 2 >$ at P. Thus, the speed is $v = |\mathbf{r}_1'| = \sqrt{1^2 + 2^2 + 2^2} = 3$.

The acceleration vector is $\mathbf{a} = \mathbf{r}_1'' =< 0, 2, 2 >$ for all points, including P. Its tangential component is its projection onto $\mathbf{r}_1'$:

$$a_T = \mathbf{a} \cdot \mathbf{r}_1'/v = (0 \times 1 + 2 \times 2 + 2 \times 2)/3 = \frac{8}{3}.$$

Its normal component is, then:

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{8 - 64/9} = \frac{2\sqrt{2}}{3}.$$

The student could have used the formulas on p. 911 to get a_N and a_T from $\mathbf{r}_1'$ and $\mathbf{r}_1''$. Or they could compute the normal vector to the curve. But the above seems to be the easiest, and uses only basics about vectors from Chapter 13. For your reference, the local tangent and normal vectors are $\mathbf{T} =< 1/3, 2/3, 2/3 >$ and $\mathbf{N} =< -2\sqrt{2}/3, 1/3\sqrt{2}, 1/3\sqrt{2} >$.

Problem 3

(a) Find and sketch the domain of the function $f(x, y) = \arctan(\sqrt{x + y - 2})$.

(b) Consider

$$f(x, y) = \frac{(x-1)(y-2)}{(x-1)^2 + (y-2)^2}.$$

Compute $\lim_{(x,y)\to(1,2)} f(x, y)$ if this limit exists, or show that the limit does not exist.

(c) Let $f(x, y) = x^2 \sin(x + y^2)$. Compute f_x, $\frac{\partial f}{\partial y}$, and f_{xy} at an arbitrary point (x, y), and f_{yx} at $(0, 2)$.

Solution:

(a) The domain of f is $D = \{(x, y) : x \in \mathbb{R} \text{ and } y \geq -x + 2\}$.

(b) The limit does not exist. To see this, let $(x, y) \to (1, 2)$ along lines passing through the point $(1, 2)$, i.e., set $y - 2 = m(x - 1)$ with arbitrary $m \in \mathbb{R}$: Along $y = m(x - 1) + 2$,

$$f(x, y) = \frac{m(x-1)^2}{(x-1)^2(1+m^2)} \to \frac{m}{1+m^2} \quad \text{as } (x, y) \to (1, 2) \text{ along } y = m(x-1) + 2.$$

Clearly, the value of the limit depends on m, so the limit does not exist.

(c) We get

$$\begin{aligned}
f_x &= 2x\sin(x + y^2) + x^2\cos(x + y^2) \\
\frac{\partial f}{\partial y} &= 2x^2 y\cos(x + y^2) \\
f_{xy} &= 4xy\cos(x + y^2) - 2x^2 y\sin(x + y^2) \\
f_{yx}(0, 2) &= f_{xy}(0, 2) = 0
\end{aligned}$$