Review topics for the final exam for Math 200

This is a rough list of the things you should definitely be able to do. I think the best way to use this review sheet is to try to recall or come up with an example for each concept or technique listed below. Then try to solve that example, and if you cannot (or could not come up with one to begin with), then re-read the corresponding section in the book. It is very important to try to think of all these concepts without opening the book, at first. Copies of this review sheet will not be allowed at the exam. Good luck!

1. Vectors, and equations of lines and planes

- Two ways to write a vector: $\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.
- Basic operations with vectors: addition, subtraction, scalar multiplication.
- Finding a unit vector parallel to a given vector $\mathbf{a}$: $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$.
- Dot product and cross product of vectors.
- The scalar component of $\mathbf{a}$ along $\mathbf{b}$; vector projection of $\mathbf{a}$ onto $\mathbf{b}$.
- parametric and symmetric forms of the equation of a line in space (how to write an equation of the line that contains two given points). How to determine whether two lines in space intersect.
- Equations of planes: normal vector; how to write an equation of the plane through three given points.
- Finding distances: between two points, from a point to a plane, from a point to a line.
- How to find symmetric and parametric equations for the line of intersection of two planes.

2. Quadric surfaces and cylinders

This section is not explicitly on the exam. However, you need to be able to sketch things like cones, spheres, and circular paraboloids, since they are often used in problems about integration. Also, remember, that a "cylinder" means any surface that consists of lines parallel to a given line and passing through points on a given plane curve (so a cylinder doesn't have to be a circular cylinder).

3. Partial derivatives for functions of two or three variables (Sections 12.3 – 12.8 of the main text and Section 14.8 of Secondary Text No. 1, except for two constraints)

- Partial derivatives.
- Linearizations: Remember that the linearization of the function $f(x, y)$ at the point (a, b) is $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$, and its graph $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is the tangent plane to the graph of the function $f(x, y)$ at the point (a, b).
- Using the linearization (or "differentials") to find the approximate value of the function, or the approximate error, etc.
- Chain rule (for functions of any number of variables).
- Implicit differentiation. Suppose z is an implicit function of x and y defined via some relation $F(x, y, z) = 0$. (if the right-hand side is not zero, take it all to the left). You need to find $\partial z / \partial x$ and $\partial z / \partial y$. To do that, either

memorize the formulas:
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}; \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z},$$
and know how to use them. Or, to find $\partial z/\partial x$, simply differentiate both sides of the relation defining $f(x, y)$ with respect to x, and every time you see z, treat it as a *function* of x and y (so use chain rule, product rule, etc.). Then you'll get a relation involving the unknown $\partial z/\partial x$. Then you can solve for it (this is exactly the way to arrive at the formula above). Please note that z can appear in the expressions you get for $\partial z/\partial x$ and $\partial z/\partial y$. To evaluate them at a given point, usually you need all three coordinates x, y, z.

- Directional derivatives; the gradient vector. The main formula:
$$D_{\bar{u}} f = \nabla f \cdot \bar{u}.$$
Remember that $\bar{u}$ in this formula has to be a unit vector. If the vector defining the diretion is not unit, you have to first make it unit. You need to understand the meaning (in words) of the directional derivative – the rate of change of f in the direction of $\bar{u}$.

- Because of the above formula, the direction of the fastest increase of the function is given by the gradient, and the direction of the fastest decrease is opposite to the gradient.

- contour maps (and the fact that at every point, gradient is perpendicular to the level curve through this point).

- You need to know how to use the gradient vector of a function of 3 variables to write an equation of the tangent plane to a surface given by some equation in x, y, z-variables.

    Note that we have two ways of writing the equation of a tangent plane to a graph of $f(x, y)$ (As a surface in $\mathbb{R}^3$, such a graph is given by the equation $z = f(x, y)$). The first way is using linearization of f (see above). The second way is to use the gradient of a function of 3 variables: You can write $z = f(x, y)$, and make a new function $F(x, y, z) = f(x, y) - z$. Then the graph is the level surface given by $F(x, y, z) = 0$. Then you can use the gradient of F to find the normal vector to the tangent plane at a point $(a, b, f(a, b))$. A good exercise: prove to yourself that both methods of writing the equation of the tangent plane give the same plane (the equation looks a little different).

- Critical points and the second derivative test for functions of 2 variables.

- Max/min of a function on a closed bounded domain (need to know how to check if the domain is closed; find critical points inside and find max/min on the boundary).

- Lagrange multipliers (one constraint only).

4. Integration

- Iterated integrals. How to write an integral of a function of two variables over a domain in the plane, and how to change the order of integration. In particular, need to be able to recognize when you need to change the order of integration in order to be able to compute the integral.

- Double integrals in polar coordinates.

- Applications of double integrals: total mass, centre of mass, average value of a function. (depending which book you are studying from, you might see moments of inertia and applications to probability – these topics are not on the exam).
- Triple integrals: you need to be able to evaluate triple integrals if they are already given as an iterated integral; and need to know how to set up a triple integral as an iterated integral (that is, define the limits of integration correctly) for not very complicated solids, as a minimum: boxes (the fancier name for them is a rectangular prism), triangular pyramids (the official name is tetrahedron), solids bounded by cylinders and planes, etc. (to the extent of suggested problems and Webwork). You need to be able to change the order of integration in triple integrals (though maybe not in the most complicated cases).
- Triple integrals in cylindrical coordinates. This includes integration over solids such as circular paraboloids and cones, etc.
- Triple integrals in spherical coordinates. You need to be able to set up such integrals correctly over pieces of spheres and cones (as in suggested problems and webwork), and to switch between spherical and Cartesian coordinates.
- It is helpful to know how to write equations of:
  - In polar coordiantes on the plane:
    * a circle with centre at the origin
    * a circle with centre on the x-axis, or on the y-axis, and passing through the origin
    * a line through the origin
  - In cylindrical coordinates in space:
    * a cone
    * a circular paraboloid
    * a vertical plane passing through the origin
    * a sphere centred at the origin.
  - In spherical coordinates:
    * cones around the z-axis
    * spheres centred at the origin
    * spheres with centre on the z-axis and passing through the origin
    * planes containing the z-axis
    * horizontal planes

Remember that when in doubt, you can take the equation in Cartesian coordinates, and simply use the conversion formulas to convert it to cylindrical or spherical.