MATH 200:921, Quiz 5

First Name: ______________________ Last Name: ______________________

Student-No: ____

Grade:

- Do not turn the page until instructed to do so.
- This test is closed book. No calculators or formula sheet allowed.
- You have 20 minutes to write this quiz.
- There are three questions in this quiz, worth a total of 20 points.

Short answer question

1. $\boxed{\text{4 marks}}$ For each of the following statements write T for true or F for false next to it.

   1. We always have
   $$\int_0^1 \int_{a(x)}^{b(x)} h(x)g(y)\,\mathrm{d}y\,\mathrm{d}x = \left(\int_0^1 h(x)\,\mathrm{d}x\right)\left(\int_{a(0)}^{b(1)} g(y)\,\mathrm{d}y\right).$$

   2. If $f(x,y)$ is continuous then it is always true that
   $$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

   3. We have
   $$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} x\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\pi} \int_0^1 r^2\cos(\theta)\,\mathrm{d}r\,\mathrm{d}\theta.$$

   4. If the density function is constant, the center of mass of a region D must always lie inside of D.

Solution:

1. False, as extremes of integration on the right hand side are nonsensical and in general they need to be computed case by case.
2. True, we can always switch extremes of integration on a rectangle when the function being integrated is continuous.
3. True, as $-1 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}$ describes the upper half circle of radius one.
4. False, take for example the circular crown $\leq x^2 + y^2 \leq 2$, whose center of mass is at the origin which is not on it.

Long answer question—you must show your work

2. $\boxed{\text{8 marks}}$ Consider the integral

$$\int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

1. Sketch the domain of integration and rewrite the integral as a $\mathrm{d}x\,\mathrm{d}y$ integral.
2. Evaluate the integral when $f(x,y) = e^{8y-\frac{2}{3}y^3}$.

Solution:

1. The boundaries of our domain are the segment $-2 \le y \le 2, x = 0$ and the parabola $x = 4 - y^2$, obtained by squaring the equation $y = \pm\sqrt{4-x}$. Then the integral becomes
$$\int_{-2}^{2}\int_{0}^{4-y^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$
2. Integrating the function $e^{8y-\frac{2}{3}y^3}$ seems hard if not to impossible to integrate right away, so we try integrating over x first:
$$\int_{-2}^{2}\int_{0}^{4-y^2} e^{8y-\frac{2}{3}y^3}\,\mathrm{d}x\,\mathrm{d}y = \int_{-2}^{2} e^{8y-\frac{2}{3}y^3}\,[x]_0^{4-y^2}\,\mathrm{d}y = \int_{-2}^{2}(4-y^2)e^{8y-\frac{2}{3}y^3}\,\mathrm{d}y.$$
Now note that $(8y - \frac{2}{3}y^3)' = 2(4-y^2)$, so by substitution we get
$$\int_{-2}^{2}(4-y^2)e^{8y-\frac{2}{3}y^3}\,\mathrm{d}y = \int_{-16+\frac{16}{3}}^{16-\frac{16}{3}} \frac{1}{2}e^u\mathrm{d}u = \frac{e^{\frac{32}{3}} - e^{-\frac{32}{3}}}{2}.$$

Long answer question—you must show your work

3. $\boxed{\text{8 marks}}$ Consider the triangle T with vertices $A = (0, 0), B = (1, 1), C = (1, 0)$.

   1. Sketch T and describe the side $\overline{CB}$ with polar coordinates equations $r = a(\theta), c \leq \theta \leq d$.
   2. *Using an integral in polar coordinates*, compute the area of the triangle.
   3. Assuming that the mass distribution on T is a constant ρ, write integrals in polar coordinates that compute the coordinates of the center of mass of T. You do not need to evaluate them.

Solution:

1. The vertical side $\overline{BC}$ of the triangle is described by $x = 1, 0 \leq y \leq 1$. The equation $x = 1$ can be rewritten $r\cos(\theta) = 1$ which gives us $r = \sec(\theta)$. The slopes of the remaining sides are 0 and 1, so θ varies between $0 = \arctan(0)$ and $\pi/4 = \arctan(1)$.
2. Using the description above (note that a radius coming out from the origin wth angle θ meets the boundary at $r = 0$ and $r = \sec(\theta)$ we get the integral

$$\int_0^{\pi/4} \int_0^{\sec(\theta)} r \mathrm{d}r \mathrm{d}\theta$$

   where as usual the r comes from $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$. Solving the integral we get

$$\int_0^{\pi/4} \frac{1}{2}\sec^2(\theta)\mathrm{d}\theta = \frac{1}{2}\left[\tan(\theta)\right]_0^{\pi^4} = \frac{1}{2}.$$

3. Knowing that the area is $\frac{1}{2}$ and that the density is constant, the integrals are

$$\overline{x} = 2\int\int_T x\mathrm{d}A, \overline{x} = 2\int\int_T y\mathrm{d}A$$

   which in polar coordinates become

$$\overline{x} = 2\int_0^{\pi/4} \int_0^{\sec(\theta)} r^2\cos(\theta)\mathrm{d}r\mathrm{d}\theta$$

   and

$$\overline{y} = 2\int_0^{\pi/4} \int_0^{\sec(\theta)} r^2\sin(\theta)\mathrm{d}r\mathrm{d}\theta$$

   as $x = r\cos(\theta), y = r\sin(\theta)$.