MATH 200:921, Quiz 4

First Name: ____________________ Last Name: ____________________

Student-No: ____

Grade:

- Do not turn the page until instructed to do so.
- This test is closed book. No calculators or formula sheet allowed.
- You have 20 minutes to write this quiz.
- There are three questions in this quiz, worth a total of 20 points.

Long answer question—you must show your work

1. 6 marks Consider the plane

$$H : x - y + z = 1.$$

   1. Let $O = (0, 0, 0)$ be the origin of the axes in $\mathbb{R}^3$. Given a point P in the plane H, write the square norm $\|\vec{OP}\|^2$ as a function $f(x, y)$ of two variables.
   2. Classify each critical point of $f(x, y)$ as either local maximum, local minimum, saddle point or undetermined.

Solution:

1. The square norm $\|\vec{OP}\|^2$ for a general point $P = (x, y, z)$ is $x^2 + y^2 + z^2$. Plugging the equation

$$z + x - y = 1 \sim z = y - x + 1$$

into the function we get

$$f(x, y, y - x + 1) = x^2 + y^2 + (y - x + 1)^2 = x^2 + y^2 + x^2 + y^2 + 1 - 2xy - 2x + 2y$$
$$= 2x^2 + 2y^2 - 2xy - 2x + 2y + 1.$$

2. We have

$$f_x = 4x - 2y - 2, \quad f_y = 4y - 2x + 2, f_{xx} = 4, f_{yy} = 4, f_{xy} = f_{yx} = -2$$

So to get the critical points we set up the system

$$\begin{cases} 4x - 2y - 2 = 0 \\ 4y - 2x + 2 = 0 \end{cases} \sim \begin{cases} 2x + 2y = 0 \\ 6y + 2 = 0 \end{cases} \sim \begin{cases} x = \frac{1}{3} \\ y = -\frac{1}{3} \end{cases}$$

So there is a single critical point which is a local minumum as

$$D\left(\frac{1}{3}, -\frac{1}{3}\right) = \begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix} = 12 > 0$$

and $f_{xx}(\frac{1}{3}, -\frac{1}{3}) = 4 > 0$.

Long answer question—you must show your work

2. 8 marks On the plane H from the previous question consider the triangle T whose vertices are

$$A = (1, 2, 2), B = (1, -1, -1), C = (-2, -1, 2).$$

Let T' be the projection of T on the xy plane.

1. Sketch T' and write down its sides and vertices. A side should be described by an equation and a range, such as $y = 2x, 0 \leq x \leq 2$ or $x = 3, 1 \leq y \leq 5$.
2. Make a list of the points in the segment $\overline{AB}$ which could realize the minimum or maximum distance from the origin. You do not need to evaluate the function at these points.
3. If you wanted to find the minimum and maximum distance from a point of T to the origin, at how many points in T' would you need to evaluate the distance function at most? Explain your reasoning.

Solution:

1. The triangle T' has vertices

$$A' = (1, 2), B' = (1, -1), C' = (-2, -1).$$

By simple observation we see that the sides are

$$x = 1, -1 \leq y \leq 1, \quad y = -1, -2 \leq x \leq 1, \quad y = x + 1, -2 \leq x \leq 1.$$

2. As the triangle lies on H, minimizing the distance from the origin to a point of T is equivalent to minimizing the function $f(x, y) = 2x^2 + 2y^2 - 2xy - 2x + 2y + 1$ on T'. We first proceed to plug the equation for the side into f:

$$x = 1, -1 \leq y \leq 1, \quad f(1, y) = 2 + 2y^2 - 2y - 2 + 2y + 1 = 2y^2 + 1$$

Then $f(1, y)' = 4y$ and the only critical point is at $(1, 0)$ which lies on T'. The corresponding point on T is $(1, 0, 0)$. Finally we also need to add the vertices A and B, so our list is

$$(1, 2, 2), (1, -1, -1), (1, 0, 0).$$

3. In the previous question we have seen that the only critical point for $f(x, y)$ is $(\frac{1}{3}, -\frac{1}{3})$ which lies on T'. Additionally we will have to check each side, where the equation $f' = 0$ has at most one solution as it is linear, and the three vertices, for a total of 7 points.

Long answer question—you must show your work

3. 6 marks Consider the cylinder given in three dimensional space by

$$S : x^2 + y^2 = 5.$$

Let R be the curve obtained by intersecting S with the plane H from the first two questions.

1. Set up the Lagrange multiplier system of the equation needed to find the points in R which minimize or maximize the square distance from the origin. Do not solve it.
2. For each of the following points, determine if it is possible or not that they minimize or maximize the square distance from R to the origin:

$$P_1 = (0, \sqrt{5}, 1 + \sqrt{5}) \quad P_2 = (2, 1, 2) \quad P_3 = (0, -1, 0)$$

Solution:

1. We want to minimize the value of $f(x, y) = 2x^2 + 2y^2 - 2xy - 2x + 2y + 1$ on the curve defined by $g(x, y) = 0$ where $g(x, y) = x^2 + y^2 + 2$. The Lagrange multiplier system is:

$$\vec{\nabla} f(x, y) - \lambda \vec{\nabla} g(x, y) = 0 \sim \begin{cases} 4x - 2y - 2 = 2\lambda x \\ 4y - 2x + 2 = 2\lambda y \\ x^2 + y^2 = 2 \end{cases}$$

2. We need to check if the points P_1, P_2, P_3 solve the system of equations above for some value of λ.

$$P_1 : \quad \begin{cases} 2\sqrt{(5)} - 2 = 0 \quad \textbf{impossible} \\ 4\sqrt{(5)} + 2 = 2\lambda\sqrt{(5)} \\ \sqrt{5}^2 = 5 \quad \textbf{ok} \end{cases} \qquad P_2 \quad \begin{cases} 4 = 4\lambda \Rightarrow \lambda = 1 \\ 2 = 2\lambda \quad \textbf{ok} \\ 2^2 + 1^2 = 5 \quad \textbf{ok} \end{cases}$$

$$P_3 \quad \begin{cases} 0 = 0 \quad \textbf{ok} \\ -2 = -2\lambda \Rightarrow \lambda = 1 \\ 0^2 + (-1)^2 = 5 \quad \textbf{impossible} \end{cases}$$

So P_2 is the only candidate.