MATH 200:921, Quiz 3

First Name: ______________________ Last Name: ______________________

Student-No: ______________________

Grade:

- Do not turn the page until instructed to do so.
- This test is closed book. No calculators or formula sheet allowed.
- You have 20 minutes to write this quiz.
- There are three questions in this quiz, worth a total of 20 points.

Long answer question—you must show your work

1. 6 marks 1. Find the domain of the function $f(x, y) = \log(y) - \sqrt{y - 1 - x^2}$ and sketch it.

2. Find a vector parametric equation for the tangent line to the trace of the graph of $f(x, y)$ on the plane $x = 0$ at the point $(0, 1, 0)$.

Solution:

1. We have two constraints on our domain. First, for $\log(y)$ to be defined we need $y > 0$. Second, for $\sqrt{y - 1 - x^2}$ to be defined we need $y \geq x^2 + 1$. The second condition also implies that $y \geq 1$, making the first redundant. Then the domain is given by $y \geq x^2 + 1$, which is the set of points on or above an upward parabola whose minimum is at $(0, 1)$.
2. If compute the partial derivative with respect to y we get $f_y = \frac{1}{y} - \frac{1}{2\sqrt{y-1-x^2}}$ which is not defined at $(0, 1)$. Intersecting with $x = 0$ we get the function $f(0, y) = \log(y) - \sqrt{y - 1}$ whose derivative goes to negative infinity for y going to 1, which means that the tangent line to the curve is vertical. Then the line we are looking for is
$$\vec{l}(t) = \langle 0, 1, 0 \rangle + t\langle 0, 0, 1 \rangle.$$

Long answer question—you must show your work

2. 6 marks Let $f(x,y,z) = e^y x + e^z y$ and let $x(u,v) = u^2, y(u,v) = uv, z(u,v) = v^2$.
Compute the partial derivatives

$$\frac{\partial f(u,v)}{\partial u}\mid_{(1,2)}, \frac{\partial f(u,v)}{\partial v}\mid_{(1,2)}.$$

Solution: We have $x(1,2) = 1$, $y(1,2) = 2, z = (1,2) = 4$. By the chain rule we need to compute

$$\frac{\partial f(u,v)}{\partial u}\mid_{(1,2)} = \vec{\nabla} f \mid_{(1,2,4)} \cdot \langle x_u(1,2), y_u(1,2), z_u(1,2)\rangle$$

and

$$\frac{\partial f(u,v)}{\partial v}\mid_{(1,2)} = \vec{\nabla} f \mid_{(1,2,4)} \cdot \langle x_v(1,2), y_v(1,2), z_v(1,2)\rangle.$$

We have $\langle x_u, y_u, z_u\rangle = \langle 2u, v, 0\rangle$ and $\langle x_v, y_v, z_v\rangle = \langle 0, u, 2v\rangle$.
Moreover, $f_x = e^y$, $f_y = e^y x + e^z$, $f_z = e^z y$. Plugging in the values we get

$$\vec{\nabla} f \mid_{1,2,4} = \langle e^2, e^2 + e^4, 2e^4\rangle$$

$$\frac{\partial f(u,v)}{\partial u}\mid_{(1,2)} = \langle e^2, e^2 + e^4, 2e^4 \cdot\rangle \cdot \langle 2,2,0\rangle = 4e^2 + 2e^4$$

$$\frac{\partial f(u,v)}{\partial v}\mid_{(1,2)} = \langle e^2, e^2 + e^4, 2e^4\rangle \cdot \langle 0,1,4\rangle = e^2 + 9e^4.$$

Long answer question—you must show your work

3. [8 marks] Consider the surface S defined by $e^y x + e^z y = 1$. The point $P = (0, 1, 0)$ lies on S.

- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at P.
- Use linear approximation to estimate the value of z when $x = 1.1, y = 1.05$.

Solution:

- By the chain rule applied to implicit differentiation we know that on the surface defined by $F(x, y, z) = k$ we have

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}.$$

  We can take $F(x, y, z) = e^y x + e^z y, \quad k = 1$ and looking at the previous question we get $F_x = e^y, F_y = e^y x + e^z, F_z = e^z y$. As $F_z(0, 1, 0) \neq 0$ we can plug the values into the equations above, getting

$$\frac{\partial z}{\partial x} = \frac{-e^1}{e^0 \cdot 1} = -e, \frac{\partial z}{\partial y} = \frac{-e^1 \cdot 0 - e^0}{e^0} = -1.$$

- We have to approximate the value of z starting from the point we know, that is $(a, b, c) = (0, 1, 0)$. The linear approximation of $z = z(x, y)$ around (a, b, c) is

$$L(x, y) = z(0, 1) + z_x(0, 1)(x - 0) + z_y(0, 1)(y - 1) = 0 - e \cdot x - (y - 1)$$

  so we get the value

$$z(1.1, 1.05) \approx 0 - e \cdot 1.1 - (1.05 - 1) = -11/10e - 1/20.$$

Note: if you computed the derivatives at point in the typo $(1, 1, 0)$ it's fine, the only difference is that the level curve is $F(x, y, z) = 2e$ rather than $F(x, y, z) = 1$. If you elected to approximate the value of z at the point $(0.1, 1.05)$ rather than $(1.1, 1.05)$, which is consistent with the spirit of the original question, it is fine as well.