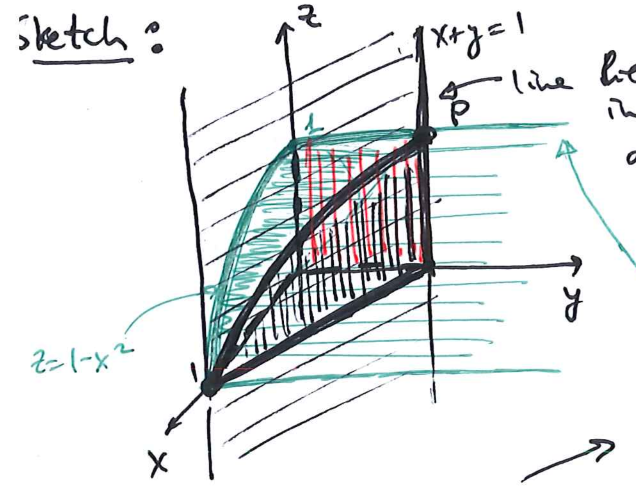
Today: Problem 2 from Worksheet 11

Recall: solid: coordinate planes, the plane $x+y=1$, $z=1-x^2$

Want: set up an integral over it as $\iiint f \, dx\, dy\, dz$

Last time: Set it up $\int_0^1\int_0^{1-x}\int_0^{1-x^2} dz\, dy\, dx$

Sketch:



Note on how to sketch.

First: emphasize all "ribs" that you see

Next: think of more "ribs" –

where does the green surface meet the plane $x+y=1$?

– parabolic cylinder meets a plane: intersection is a parabola?

What points does it contain?

$(1,0,0)$, $(0,1,1)=P$

1

Now: problem 2 we want: $\iiint \_\_ \, dx\,dy\,dz$

we have: $\int_0^1 \int_0^{1-x} \int_0^{1-x^2} \_\_ \, dz\,dy\,dx$

From the set-up we have:

$0 \leq x \leq 1$

$0 \leq y \leq 1-x$

$0 \leq z \leq 1-x^2$

max possible z

want: $? \leq z \leq ?$ ← numbers

rewrite boundaries for x, y in terms of z.

min possible z = 0 - easy

Max possible z: $z \leq 1-x^2 \leq 1$, so definitely $z \leq 1$.

is $z=1$ possible? - Yes.
(sketch; plug in $x=0$).

So: $\underbrace{\int_0^1 \quad\quad dz}_{\text{our outside integral.}}$

Now: want limits for y in terms of z.
(cannot use x!!)

$$0 \le y \le \underbrace{1-x}.$$

↑ how big can this be?

we also have: $z \le 1-x^2$

$$x^2 \le 1-z$$

$$\underbrace{-\sqrt{1-z}}_{\text{not relevant.}} \le x \le \underline{\sqrt{1-z}} \qquad \text{also } \underline{0 \le x}.$$

So: the biggest $1-x$ is still 1:

Got: $1-x \le 1$,

Looks like we got: $\displaystyle\int_0 \int_0^1 dy\, dz$

Does this make any sense?
we got these bounds when $x=0$.
Look at the cross-section by the plane $x=0$.
This agrees with the picture! ← Red square (see sketch on p. 1)

Continue: $\int_0^1 \int_0^1 \quad dy\, dz$

What are the limits for x? (can use z, y).

We should have:

$0 \leq y \leq 1-x \qquad x \leq 1-y$

$0 \leq z \leq 1-x^2 \qquad x^2 \leq 1-z$

$x \leq \sqrt{1-z}$

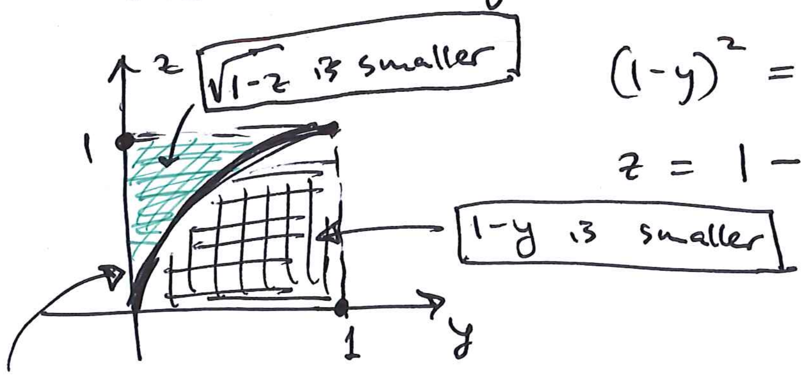
both have to hold. (they are competing!)

Which one wins? (is smaller)

when is $1-y = \sqrt{1-z}$? (all is positive b/c $1 \geq y \geq 0$, $1 \geq z \geq 0$)

$(1-y)^2 = 1-z$

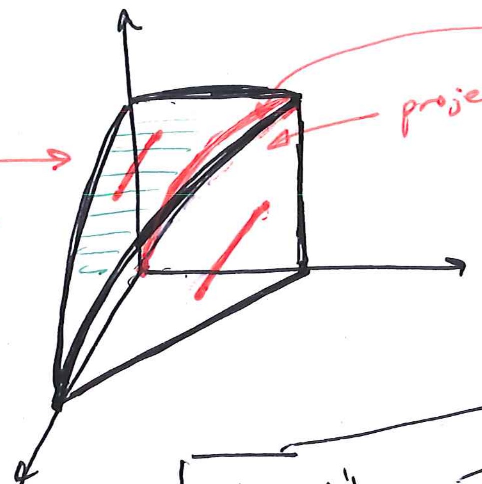
$z = 1-(1-y)^2$



$y = 1 - \sqrt{1-z}$

So the integral over the red square (in yz-variables) breaks into two pieces:

$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} \sqcup \, dx\, dy\, dz \;+\; \int_0^1 \int_{1-\sqrt{1-z}}^{1} \int_0^{1-y} \sqcup \, dx\, dy\, dz$$



projection of the rib onto yz-plane: this is the curve that divides the square

for each (y,z) in the square in the back, the integral dx is on the piece of the line parallel to the x-axis that lies inside the solid.

"roof" is $x^2 = 1-z$

the line parallel to the x-axis exits the solid "through the roof" in the & green part of the square; through the side wall: $y = 1-x$ in the black part.

<u>Note</u>: Setting up the integral in the other order:

$\iiint \sqcup \, dy\, dx\, dz$ would be easier again:

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} \sqcup \, dy\, dx\, dz$$

(the limits for z we figured out already; we also figured out $0 \le x \le \sqrt{1-z}$; and the <u>only</u> constraint on y is $0 \le y \le 1-x$

Today: Cylindrical (and spherical?) coordinates.

Example: solid is between two paraboloids:

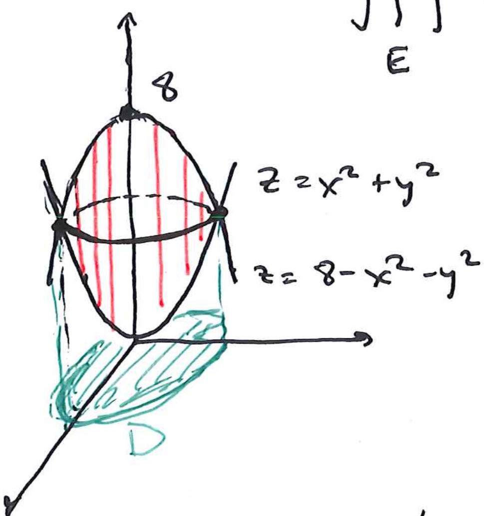
$$z = 8 - x^2 - y^2, \qquad z = x^2 + y^2.$$

Find the z-coordinate of its centroid.

Solution: take density $\equiv 1$.

$$M = V = \iiint_E 1 \, dV = \iint_D \left(\int_{z = x^2+y^2}^{z = 8 - x^2 - y^2} 1 \, dz \right) dA$$

↑ integrates on ||||



$$= \int_0^{2\pi} \int_0^2 \int_{r^2}^{8 - r^2} (1 \, dz) \, r \, dr \, d\theta$$

↑ convert to polar on D

↑ do not forget to convert expressions for the limits!

Need to find the radius of the green circle.

$x^2 + y^2 = 8 - x^2 - y^2$ – on the circle of intersection.

(see p. 2 for the continuation of finding this radius)

$x^2+y^2 = 8-x^2-y^2$

$2(x^2+y^2) = 8$

$x^2+y^2 = 4$ ← also $z=4$. ← these two equations define the circle of intersection.

Now: project onto xy-plane (forget z).

$x^2+y^2=4$ ← boundary of D.

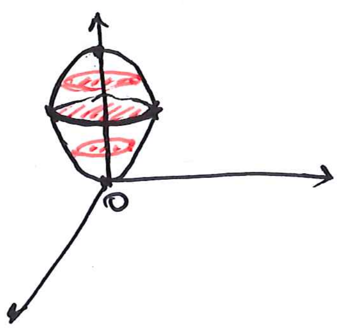
This integral evaluated and $\bar{z}$ found on p. 7

<u>Get</u>: $M = V = \int_0^{2\pi}\int_0^{2} \underbrace{\int_{r^2}^{8-r^2} 1\,dz}_{=\,8-2r^2} \underbrace{r\,dr\,d\theta}_{\text{from integral on } D} = \boxed{\text{see below} ☺ \ \text{← p. 7} \\ \ldots \quad \frac{8\pi}{3}\,? \quad \frac{16\pi}{3}\,?}$

↑ suggested answers.

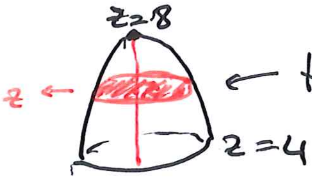
How to do this in another order?

Think of it as a double integral inside a single integral:



$$\int_0^8 \Big(\underbrace{\iint}_{\substack{\text{red disc} \\ \text{(depends on } z)}} 1\,dA \Big)\,dz$$

↑ need to find its radius



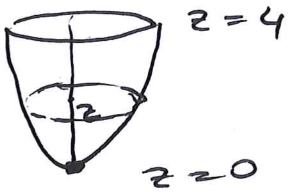
← top half

$z = 8 - x^2 - y^2$

$r^2 = x^2 + y^2$ ← radius of our disc

$r^2 = 8 - z$, $r = \sqrt{8-z}$ $(r > 0!)$

bottom half: $r = \sqrt{z}$



Get: $M = \underbrace{\int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} 1 \, r \, dr \, d\theta \, dz}_{\text{bottom half}} + \underbrace{\int_4^8 \int_0^{2\pi} \int_0^{\sqrt{8-z}} 1 \, r \, dr \, d\theta \, dz}_{\text{top half}}$

bottom half:

$0 \le z \le 4$

radius of cross-section at height=z → $r = \sqrt{z}$

top half

$4 \le z \le 8$

$r = \sqrt{8-z}$

This is called <u>cylindrical coordinates</u>:

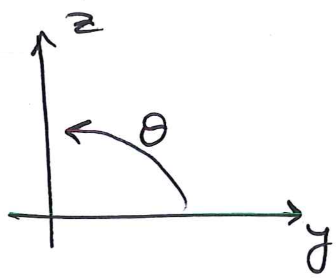
$(x, y, z) \rightsquigarrow (r, \theta, z)$

$x = r\cos\theta$

$y = r\sin\theta$

$$\boxed{dV = r\,dr\,d\theta\,dz}$$

Similarly, can convert (y, z) or (x, z) to polar:



$z = r\sin\theta$

$y = r\cos\theta$

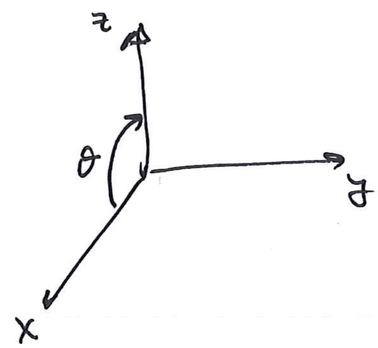
$x \leftarrow$ unchanged

$dV = r\,dr\,d\theta\,dx.$

Convert (x, z) to polar:

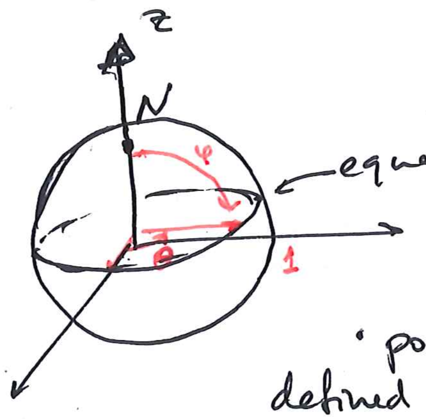
$x = r\cos\theta$

$z = r\sin\theta.$



Spherical coordinates

- good for: spheres, cones, 'wedges of spheres'.



θ = angle in the horizontal plane (from positive x-axis)

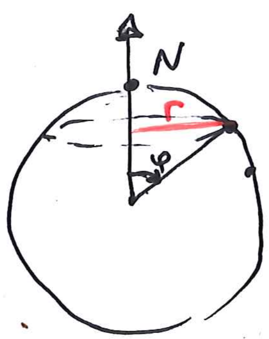
φ = angle from positive z-axis

- point in space:

defined by

- distance from $(0,0,0)$
- latitude $0 \le \varphi \le \pi$
- longitude $0 \le \theta \le 2\pi$

(Note: in geography, $-\pi/2 \le \varphi \le \pi/2$)



$\varphi = 0$ at North pole $(0,0,1)$

$\varphi = \pi/2$ at the equator

$\varphi = \pi$ at the South pole

r = radius at latitude φ.

$\boxed{r = \rho \sin\varphi}$

Conversion formulas

spherical coordinates:

(ρ, φ, θ)
$\nearrow$ radius

$\rho = \sqrt{x^2+y^2+z^2}$ - distance from $(0,0,0)$.

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

$$\boxed{dV = \rho^2 \sin\varphi \, d\varphi \, d\theta \, d\rho.}$$

$\leftarrow$ will explain next time.

Evaluation of the integral and centre of mass from Problem 1:

$$M = \int_0^{2\pi}\int_0^2 \int_{r^2}^{8-r^2} 1\,dz\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 z\Big|_{r^2}^{8-r^2} \cdot r\,dr\,d\theta$$

$$= 2\pi \cdot \int_0^2 (8-r^2-r^2)\cdot r\,dr = 2\pi \int_0^2 8r - 2r^3\,dr$$

$$= 16\pi \cdot \frac{1}{2} r^2 \Big|_0^2 - 4\pi \cdot \frac{r^4}{4}\Big|_0^2 = 32\pi - 16\pi = 16\pi.$$

Centre of mass (the z-coordinate):

$$\bar{z} = \frac{1}{M}\int_0^{2\pi}\int_0^2\left(\int_{r^2}^{8-r^2} z\cdot 1\,dz\right) r\,dr\,d\theta$$

$$= \frac{1}{M}\int_0^{2\pi}\int_0^2 \frac{z^2}{2}\Big|_{r^2}^{8-r^2} \cdot r\,dr\,d\theta$$

$$= \frac{2\pi}{2M}\int_0^2 \underbrace{\left((8-r^2)^2 - r^4\right)}_{\substack{\shortparallel \\ 64-16r^2+r^4-r^4}} \cdot r\,dr = \frac{\pi}{M}\left(64\,\frac{r^2}{2}\Big|_0^2 - 16\,\frac{r^4}{4}\Big|_0^2\right)$$

$$= \frac{\pi}{M}\cdot(128-64) \underset{\substack{\uparrow \\ \text{plug} \\ \text{in} \\ M=16\pi \text{ from above}}}{=} \frac{64\pi}{16\pi} = 4$$

Answers : $M = 16\pi$, $\bar{z} = 4$

Reality check: the solid is symmetric about the plane $z=4$, so we could have guessed that its centre of mass has to lie on this plane.

Note: when computing the mass (volume), some people just computed the mass of the bottom part and multiplied by 2. This is correct, but this doesn't work when computing $\bar{z}$ (because now you are integrating the function z over this solid, and its values are not the same on the top and on the bottom).