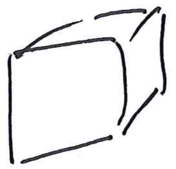
Triple integrals



solid E (filled in 3-dim object).

$f(x,y,z)$ - function of 3 variables defined on E

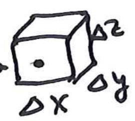
(examples: $\rho(x,y,z)$ - density of the material

$T(x,y,z)$ - temperature, solid = all the air the room / with

Define $$\iiint_E f(x,y,z)\, dV$$
↑ with respect to volume.

definition: using Riemann sums:

chop E (approximately) into small boxes



$(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ pick a point in each box

$$\sum_{i,j,k} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta_i x\, \Delta_j y\, \Delta_k z$$

↑ Riemann sum for our integral.

$\Delta_i x\, \Delta_j y\, \Delta_k z$ = volume of the box number (i,j,k)

(when f is continuous)
Riemann sums have a limit as the boxes get smaller
the limit is called $$\iiint_E f(x,y,z)\, dV.$$

To compute: set up as an iterated integral

(!) The shape of E is encoded in the limits of integration.

Example: Find the average of $f(x,y,z) = x+y-z$ over the tetrahedron bounded by the coordinate planes and the plane $3x+2y+z=6$

Note: average of a function:

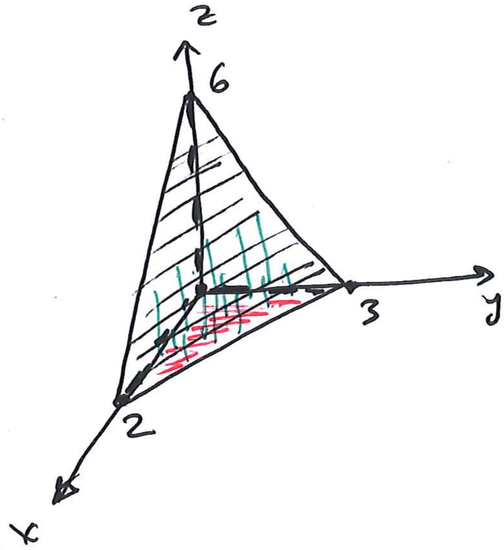
- over a 3d solid: $\frac{1}{Vol(E)} \cdot \iiint_E f(x,y,z)\,dV$

- over a 2d region: $\frac{1}{area(D)} \iint_D f(x,y)\,dA$

- over an interval $[a,b]$: $\frac{1}{b-a} \int_a^b f(x)\,dx$.

1. Volume of E: 2 ways: 1) think of E as the solid under the graph of $z = h(x,y)$

or 2) $\iiint_E 1 \, dV$.



Drawing planes:

$3x + 2y + z = 6$

- look for intercepts with axes
- think of whether the plane is parallel to something.

$3x = 6$ (x-axis)

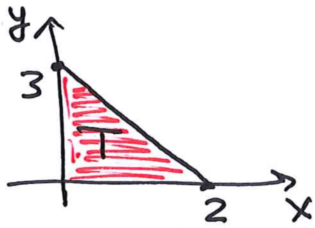
$2y = 6$

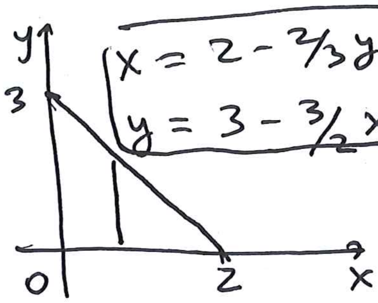
$z = 6$

Our plane: graph of $z = 6 - 2y - 3x$.

- volume: our solid: under the graph of $z = 6 - 2y - 3x$

over the triangle





$$\begin{cases} x = 2 - \tfrac{2}{3}y \\ y = 3 - \tfrac{3}{2}x \end{cases}$$

$$V = \iint_T (6-3x-2y)\, dA$$

$$= \int_0^2 \int_0^{3-3/2\,x} (6-3x-2y)\,dy\,dx$$

$$\left(= \int_0^3 \int_0^{2-2/3\,y} (6-3x-2y)\,dx\,dy \right)$$

If we were computing the volume by a triple integral,

$$V = \iint \int_0^{\text{plane}} 1\, dz = \int_0^2 \int_0^{3-3/2\,x} \int_0^{6-3x-2y} 1\, dz\,dy\,dx$$

(should be the red triangle T)

$$= \int_0^2 \int_0^{3-3/2\,x} (6-3x-2y)\,dy\,dx \quad \text{as above}$$

↑ evaluate from inside out

We have our integral:

$$\int_0^2 \int_0^{3-3/2x} \int_0^{6-3x-2y} 1 \, dz \, dy \, dx$$

outside limits have to be numbers

can use x for limits inside the integral with respect to x.

inside an integral with respect to x, y so can use x, y for limits

- For the average:

$$\text{Average}(f) = \frac{1}{V} \int_0^2 \int_0^{3-3/2x} \int_0^{6-3x-2y} (x+y-z) \, dz \, dy \, dx$$

see above

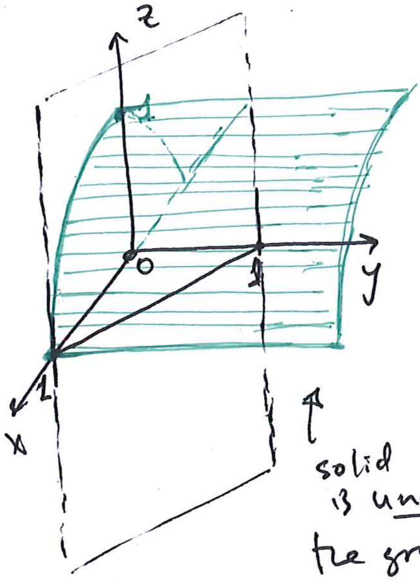
$f(x,y,z)$ from the problem

$$= \frac{1}{V} \int_0^2 \int_0^{3-3/2x} xz + yz - \frac{1}{2}z^2 \Big|_0^{6-3x-2y} dy \, dx$$

$$= \frac{1}{V} \int_0^2 \int_0^{3-3/2x} \Big(x(6-3x-2y) + y(6-3x-2y) - \frac{1}{2}(6-3x-2y)^2\Big) dy \, dx$$

Worksheet 11 Triple integrals

1. Set up, in any order, the integral representing the x-coordinate of the centroid of the solid bounded by the planes $x=0$, $y=0$, $z=0$, $y+x=1$, and the parabolic cylinder $z=1-x^2$.



↑ parabola facing down apex at (0,1) ↑x ↑z

$z=1-x^2$ consists of lines parallel to the y-axis

(y is not in the equation)

↑ solid is under the green lid in the first octant behind the plane $y=1-x$.

so $0 \le x \le 1$.

$$M=\int_0^1\int_0^{1-x}\int_0^{1-x^2} 1\cdot dz\,dy\,dx$$

For $\bar{x}$, see p. 3

2. Change the order of integration so that the iterated integral is of the form $\iiint \_\_ \,dx\,dy\,dz$

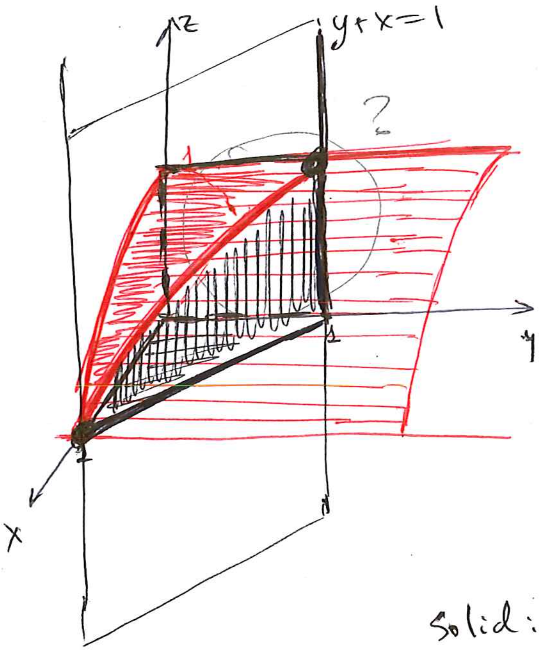
Will do problem 2 on Monday.

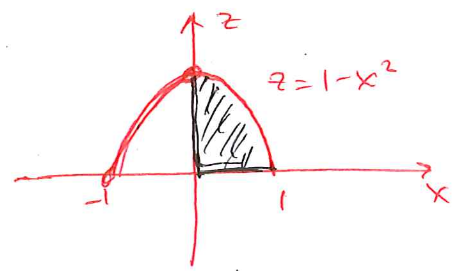
See next page for a complete picture of the solid, as a hint – think about it before Monday.

Note also that the order of integration in #1 on the next page is different from the order on this page.

Worksheet 11 — Triple integrals

1. Set up, in any order, the integral representing the x-coordinate of the centroid of the solid bounded by the planes $x=0$, $y=0$, $z=0$, $y+x=1$, and the parabolic cylinder $z=1-x^2$.





for $\bar{x}$:

$$\int_0^1\int_0^{1-x^2}\int_0^{1-x} x\cdot dy\,dz\,dx$$

Note: can switch dz dy with no effort.

Get, $0 \le x \le 1$

$$M = \int_0^1\int_0^{1-x^2}\int_0^{1-x} 1\cdot dy\,dz\,dx$$

solid: under the cylinder $z=1-x^2$

so $0 \le z \le 1-x^2$

$0 \le y \le 1-x$

2. Change the order of integration so that the iterated integral is of the form $\iiint \_\_ \, dx\,dy\,dz$

Note for #1: we notice that limits for y, z depend only on x. So if we fix x, limits for y, z are constant. Then cross-sections of our solid parallel to the yz-plane are rectangles!

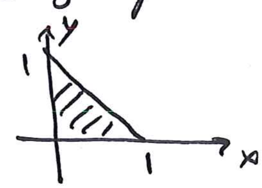
Problem 1, continued

$$\bar{x} = \frac{1}{M}\int_0^1\int_0^{1-x}\int_0^{1-x^2} x\cdot 1 \, dz\, dy\, dx, \quad \bar{y} = \frac{1}{M}\int_0^1\int_0^{1-x}\int_0^{1-x^2} y\cdot 1\, dz\, dy\, dx$$

$$\bar{z} = \frac{1}{M}\int_0^1\int_0^{1-x}\int_0^{1-x^2} 1\cdot \underline{\underline{z}} \, dz\, dy\, dx$$

Note: in this example, could say that

M = volume under the graph of $z = 1-x^2$ over



and set it up as

$$\int_0^1\int_0^{1-x} (1-x^2)\, dy\, dx$$

but say, z-coordinate of the centre of mass is

$$\frac{1}{M}\int_0^1\int_0^{1-x}\int_0^{1-x^2} \underline{\underline{z}}\, dz\, dy\, dx$$ cannot be obtained from such double integral.