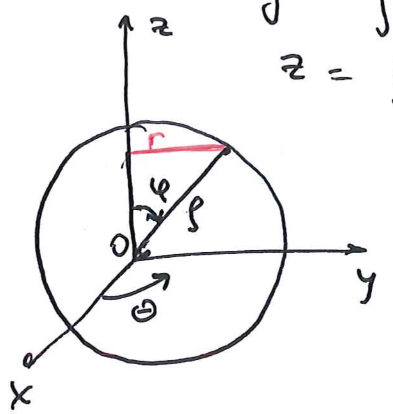
Spherical coordinates

Recall:

$$x = \rho \sin\varphi \cos\theta$$
$$y = \rho \sin\varphi \sin\theta$$
$$z = \rho \cos\varphi$$

$$r = \sqrt{x^2+y^2} = \rho \sin\varphi$$



What about volume?

$$\text{"}dV\text{"} = \rho^2 \sin\varphi \underbrace{d\rho\, d\varphi\, d\theta}_{\text{in any order}}\text{"}$$

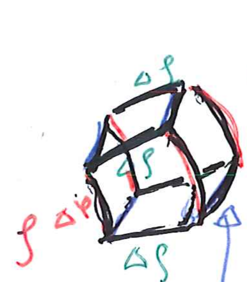
$dxdydz$

Put it all together:
volume of our piece

$$\text{is} \approx \Delta\rho \cdot \rho \sin\varphi \Delta\theta \cdot \rho \Delta\varphi$$
$$= \rho^2 \sin\varphi\, \Delta\rho\, \Delta\varphi\, \Delta\theta$$

Why the volume factor?

"spherical wedges":



arc of a circle of radius ρ corresp. to the angle $\Delta\varphi$

its length $\approx \rho\Delta\varphi$

in ~~[illegible]~~ horizontal circles

angular measure is $\Delta\theta$

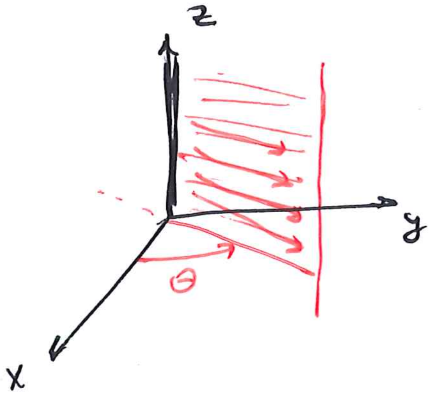
radius of these circles is $\approx \rho \sin\varphi$

length: $\approx \rho \sin\varphi \Delta\theta$

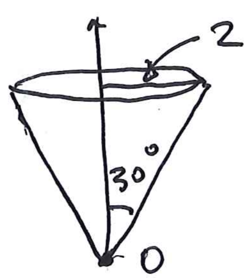
$\varphi = \text{const}$: cone

$\rho = \text{const}$: sphere

$\theta = \text{const}$: half of a vertical plane :



Example: Use spherical coordinates to find the z-coordinate of the centroid of a cone with angle $30°$ at the vertex, of radius 2 at the base.



$$\bar{z} = \frac{1}{\text{Vol(cone)}} \iiint_{\text{cone}} z \, dV$$

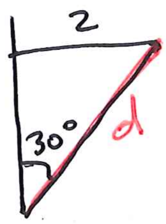
$$\mathrm{vol(cone)} = \iiint_{\text{cone}} 1 \, dV = \int_?^? \int_?^? \int_?^? \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

<u>limits</u>:

$0 \le \theta \le 2\pi$ ← cross-sections by horizontal planes are circles centred at O.

$0 \le \varphi \le 30^\circ$ (have to use radians!)
$\qquad\qquad\;\; = \pi/6$

– this gives our cone



$0 \le \rho \le d.$

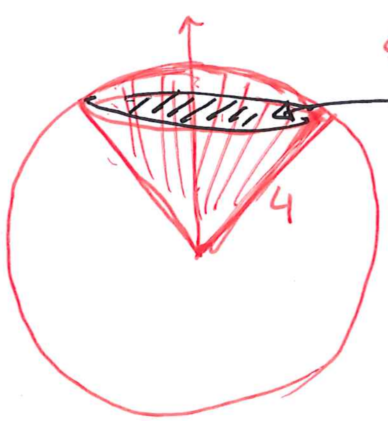
Need to find d:

$d \cdot \sin 30^\circ = 2$

$d = 4.$

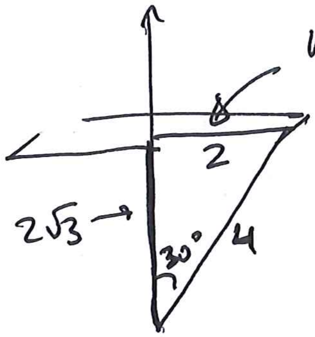
<u>Wrong thing to try:</u>

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^{4} \text{———} \, d\rho \, d\varphi \, d\theta$$



missing this cut-off by the horizontal plane.

How to correct?



need the equation of this horizontal plane in spherical coords.

The plane is given by

$$z = 2\sqrt{3}$$

$$z = \rho \cos\varphi$$

So: $\boxed{\rho\cos\varphi = 2\sqrt{3}}$

Correct integral:

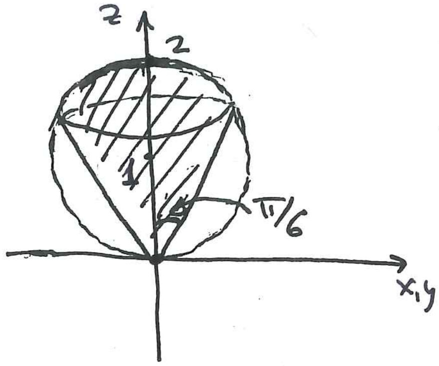
$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\sqrt{3}/\cos\varphi} 1 \, d\rho \, d\varphi \, d\theta = \text{Vol.}$$

$\boxed{\rho^2 \sin\varphi}$ ← do not forget this factor!

$$\underline{\text{Vol}} = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\sqrt{3}/\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta .$$

$$\bar{z} = \left(\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\sqrt{3}/\cos\varphi} \underbrace{\rho\cos\varphi}_{z} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \right) \cdot \frac{1}{V}$$

__Note:__ here total mass = volume because density $= 1$. (so we are using $\frac{1}{V}$ for centre of mass)



Find the z-coordinate of the centroid of the shaded region (inside the cone and inside the sphere; the cone has angle $\pi/6$ at the vertex; the sphere has radius 1.

Sphere in Cartesian coords: centre $(0,0,1)$

$$x^2+y^2+(z-1)^2=1 \qquad (*)$$

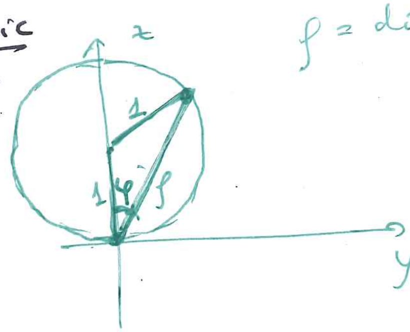
$$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$$

$x^2+y^2=\rho^2\sin^2\varphi$

Plug this into $(*)$: see next page

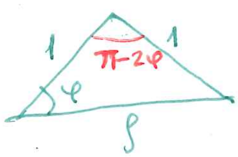
geometric methods (optional)

method #1



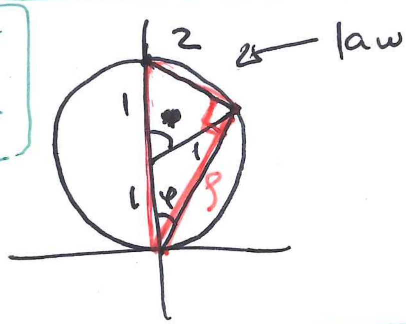
ρ = distance to our point from $(0,0,0)$

Law of cosines gives:



$$\rho^2=1^2+1^2-2\cdot1\cdot1\cos(\pi-2\varphi)$$
$$=2-2\cos(2\varphi)=4\cos^2\varphi$$

better method #2



$$\rho=2\cos\varphi$$

(d) $\underbrace{x^2+y^2}_{\rho^2\sin^2\varphi}+\underbrace{(z-1)^2}_{(\rho\cos\varphi-1)^2}=1$

$$\underbrace{\rho^2\sin^2\varphi+\rho^2\cos^2\varphi}_{\rho^2}-2\rho\cos\varphi+\cancel{1}=\cancel{1}$$

$$\rho^{\cancel{2}}=2\cancel{\rho}\cos\varphi$$

$$\boxed{\rho=2\cos\varphi}$$

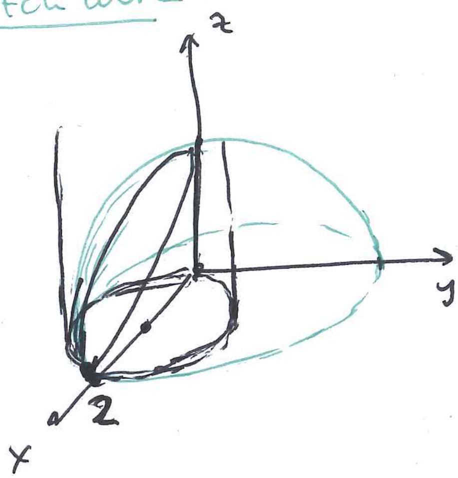
<u>Answer</u>: $\underset{\substack{\uparrow\\ \text{density}=1\\ \text{(b/c want centroid)}}}{M=V}=\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\cos\varphi}1\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

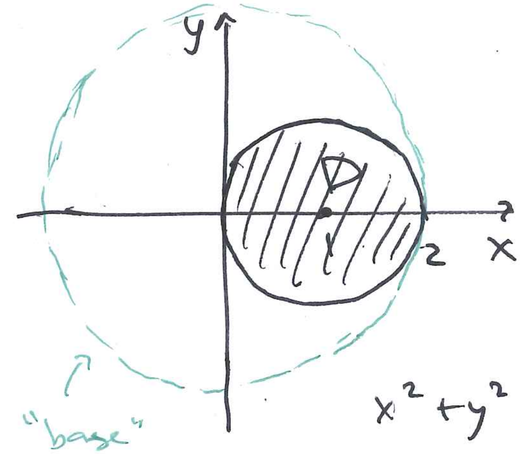
$$\bar{z}=\frac{1}{M}\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\cos\varphi}\underbrace{\rho\cos\varphi}_{z}\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

Question: Region above xy-plane, inside $x^2+y^2-2x=0$ (vertical cylinder.)

under $x^2+y^2+z^2=4$

scratch work:



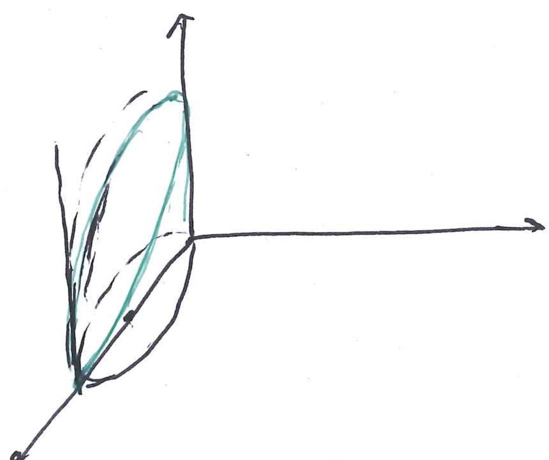


"base" of $x^2+y^2+z^2=4$.

$x^2+y^2-2x=0$

$(x-1)^2+y^2=1.$

$r=2\cos\theta$



Two partial sketches (both bad) — but not needed to set up the integral

— this is the only sketch really useful for set-up.

Set-up:

$$\iint_D \int_0^{\sqrt{4-(x^2+y^2)}} 1\cdot dV$$

disc in xy-plane

integral along z

scratch work helping us choose the coordinate system and set up the integral

cylindrical:

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$$