MULTIVARIABLE INTEGRALS: THE CONSTRUCTION

GIVEN A FUNCTION $f(x,y)$ AND A REGION R WE WANT A DEFINITION OF INTEGRAL

$$\iint_R f(x,y)\, dA$$

WHICH ONLY DEPENDS ON THE "AREA OF SMALL SQUARES" dA

<u>STEP 1:</u> $R = [a,b] \times [c,d]$, f CONSTANT $= K$

THEN

$$\iint_{[a,b]\times[c,d]} K \, dA = (b-a)(d-c)K$$

<u>STEP 2:</u> $R = [a,b] \times [c,d]$ f ANY

DIVIDE $[a,b]$ AND $[c,d]$ IN n, m INTERVALS:

$$a = x_0 < x_1 < \dots < x_n = b \qquad c = y_0 < y_1 < \dots < y_m = d$$

TAKE $x_{i-1} \le x_i^* \le x_i$ FOR $i = 1, \dots, n$, y_j^* SAME

THEN THE <u>m-TH RIEMANN SUM</u> FOR $f(x,y)$ ON R IS

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{f(x_i^*, y_j^*)}_{\text{VALUE AT SAMPLE POINT}} \underbrace{\Delta_{x,i} \Delta_{y,j}}_{\text{AREA OF } i,j \text{ TH RECTANGLE}}$$

AND WE DEFINE $\iint_R f(x,y)\,dA$ AS

$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\, \Delta_{x,i}\, \Delta_{y,j}$$

(IF $f(x,y)$ IS CONTINUOUS THE LIMIT IS DEFINED AND THE SAME FOR ALL CHOICES OF x_i^*, y_j^*)

LET'S TRY AN EXPLICIT COMPUTATION:

E.G. EXPLICITLY COMPUTE

$$\iint_{[0,1]\times[0,1]} xy\, dA$$ WITH THE DEFINITION.

WE USE EQUAL INTERVALS $x_i^* = x_i = \frac{i}{n}$, $y_j^* = y_j = \frac{j}{m}$

$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} \underbrace{\frac{i}{n}\cdot\frac{j}{m}}_{f(x_i^*, y_j^*)} \underbrace{\left(\frac{1}{nm}\right)}_{\Delta_{x,i}\Delta_{y,j}} =$$

$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \frac{j}{m^2}\left(\frac{n(n+1)}{2n^2}\right) = \lim_{m\to\infty}\left(\frac{m(m+1)n(n+1)}{4m^2n^2}\right)$$

$$= \frac{1}{4}$$

NOW, NOTE THAT UNDER SOME SUMMABILITY CONDITIONS WE HAVE

$$\lim_{n,m\to\infty} \sum_{j=1}^{m} \sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta_{x,i}\Delta_{y,j} =$$

$$= \lim_{m\to\infty}\left(\lim_{n\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} f(x_i^*, y_j^*)\,\Delta_{x,i}\,\Delta_{y,j}\right) =$$

$$= \lim_{m\to\infty}\sum_{j=1}^{m}\left(\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_j^*)\Delta_{x,i}\right)\Delta_{y,j}$$

$$= \lim_{m\to\infty}\sum_{i=1}^{m}\int_a^b f(x, y_j^*)\,dx\;\Delta_{y,j} =$$

OPPOSITE ORDER

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \cdots =$$

$$\int_a^b\int_c^d f(x,y)\,dy\,dx.$$

<u>E.G.</u> $\iint_R x^2\cos y + y^2\,dA \qquad R = [0,1]\times[0,\pi]$

$$\iint_R x^2\cos y + y^2\,dA = \int_0^1\int_0^\pi x^2\cos y + y^2\,dy\,dx =$$

$$\int_0^1\left[x^2\sin y + \frac{y^3}{3}\right]_0^\pi dx = \int_0^1 \frac{\pi^3}{3}\,dx = \frac{\pi^3}{3}$$

OPPOSITE ORDER

$$\int_0^\pi\int_0^1 x^2\cos y + y^2\,dx\,dy = \int_0^\pi\left[\frac{x^3}{3}\cos y + xy^2\right]_0^1 dy = \int_0^\pi \frac{\cos y}{3} + y^2\,dy$$

$$= \frac{\pi^3}{3}$$