MULTIVARIABLE INTEGRALS

SOME QUESTIONS:

- WHAT IS THE VOLUME OF THE REGION BETWEEN THE ELLIPTIC PARABOLOID

$$z = -2x^2 - 5y^2 + 5$$

AND THE XY PLANE?

- A THIN RECTANG. PLATE OF METAL OF SIDES 2, 4 HAS A TEMPERATURE DISTRIBUTION OF

$$T(x,y) = 2e^{-|x|-|y|} - 3$$

WHERE THE ORIGIN IS PLACED AT THE CENTER OF THE RECTANGLE.

WHAT IS THE AVERAGE TEMPERATURE ON THE PLATE?

BOTH QUESTIONS REQUIRE US TO COMPUTE THE "TOTAL SUM" OF THESE FUNCTIONS OVER A DOMAIN, THAT IS, TO FIND AN INTEGRAL.

TODAY WE'LL FIGURE OUT HOW TO COMPUTE IT, TOMORROW THE TECHNICAL DETAILS.

IDEA 1:

WE'VE SEEN HOW TO FIND AN ANTI-DERIVATIVE TO $f(x,y)$ WITH RESPECT TO A VARIABLE, THAT IS A FUNCTION

$$F(x,y) \text{ SUCH THAT } F_x(x,y) = f(x,y).$$

WE JUST TAKE SOMETHING LIKE

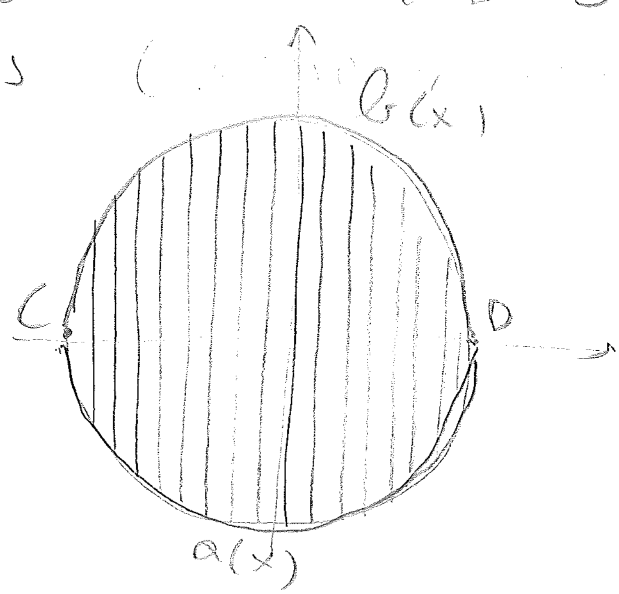
$$\int f(x,y)\,dx + C(y)$$

E.G. $f(x,y) = 2xy \quad F(x,y) = x^2 y + C(y)$

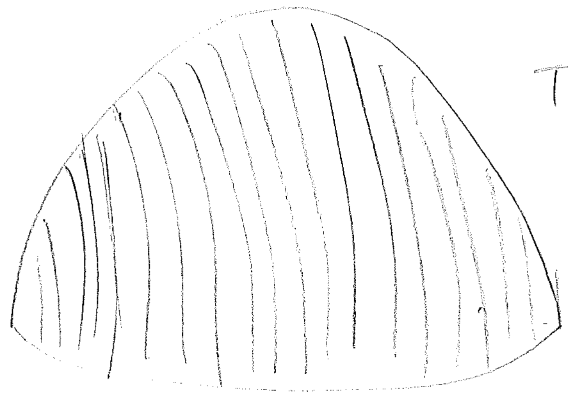
$$F(x,y) = x^2 y + \sin y \quad F_x(x,y) = 2xy$$

IDEA 2:

WE CAN CUT THE DOMAIN IN STRIPES (



AND COMPUTE THE REQUIRED QUANTITY ON EACH STRIPE

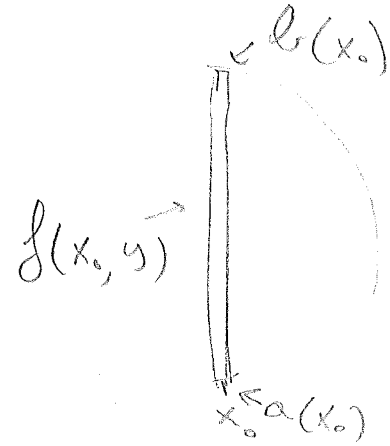


TOTAL VOL =

$\sum$ STRIPE VOL

GOING TO THE LIMIT THE VOLUME OF EACH STRIPE BECOMES

$$\left(\int_{a(x)}^{b(x)} \text{HEIGHT}(y)\, dy \right) dx$$



THEN THE SUM IS

$$\int_C^D \left(\int_{a(x)}^{b(x)} z(x,y)\, dy \right) dx$$

<u>NOTE!</u> GENERALLY

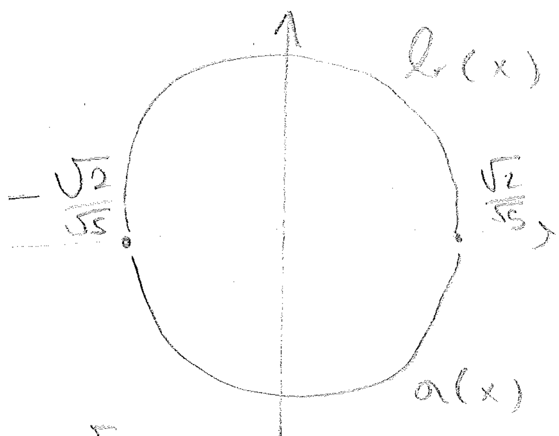
$$\int_C^D \left(\int_{a(x)}^{b(x)} z(x,y)\, dy \right) dx \neq$$

$$\int_{a(x)}^{b(x)} \left(\int_C^D z(x,y)\, dx \right) dy$$

LET'S TRY ON THE FIRST EXAMPLE

$$z(x,y) = 5 - 2x^2 - 5y^2$$

AT $z = 0$ $\quad y = \pm\sqrt{1 - \frac{2}{5}x^2}$



$$a(x) = -\sqrt{1 - \frac{2}{5}x^2}$$
$$b(x) = \sqrt{1 - \frac{2}{5}x^2}$$

$$\int_{-\frac{\sqrt{2}}{\sqrt{5}}}^{\frac{\sqrt{2}}{\sqrt{5}}} \left(\int_{-\sqrt{1-\frac{2}{5}x^2}}^{\sqrt{1-\frac{2}{5}x^2}} 5 - 2x^2 - 5y^2 \, dy \right) dx =$$

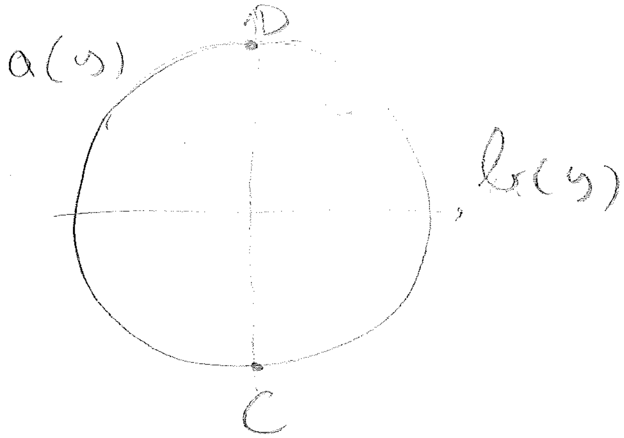
$$\int_{-\frac{\sqrt{2}}{\sqrt{5}}}^{\frac{\sqrt{2}}{\sqrt{3}}} \left[5y - 2x^2 y - \frac{5}{3} y^3 \right]_{-\sqrt{1-\frac{2}{5}x^2}}^{\sqrt{1-\frac{2}{5}x^2}} dx$$

$$= \int_{-\frac{\sqrt{2}}{\sqrt{5}}}^{\frac{\sqrt{2}}{\sqrt{5}}} 2\left(\sqrt{1-\frac{2}{5}x^2}\left(5 - 2x^2 - \frac{5}{3}\left(1 - \frac{2}{5}x^2\right)\right)\right) dx =$$

$\ldots = \frac{5}{2}\sqrt{\frac{5}{2}}\,\pi \simeq 12.418.$

ANNOYING BUT DOABLE INTEGRAL

IF WE WANTED TO INTEGRATE W.R.T. X FIRST?



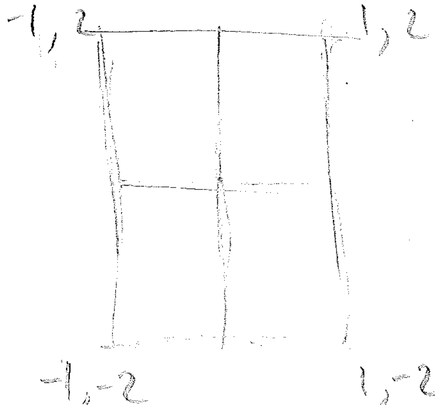
$a(y) = -\sqrt{1 - \frac{5}{2}y^2}$

$b(y) = \sqrt{1 - \frac{5}{2}y^2}$

$C = -1 \qquad D = 1$

$$\text{VOL} = \int_{-1}^{1} \int_{-\sqrt{1-\frac{5}{2}y^2}}^{\sqrt{1-\frac{5}{2}y^2}} 5 - 2x^2 - 5y^2 \, dx \, dy$$

E.G. TEMPERATURE QUESTION



$f(x,y) = 2e^{-|x|-|y|} + 3$

$f(\pm x, \pm y) = f(x,y)$

SO WE CAN RESTRICT

TO $x, y \geq 0$

AVG TEMPERATURE: RECALL

AVG OF A FUNCTION (1 VAR)

$$\underset{[a,b]}{\text{AVG}}(f) = \frac{1}{b-a}\int_D f \, dx$$

SO IT STANDS TO REASON THAT

FOR $D \subset \mathbb{R}^2$ WE WOULD HAVE

$$\underset{D}{\text{AVG}}(f) = \underbrace{\iint_D f \, dx\,dy}_{\text{WATHEVER THIS MEANS}} \cdot \frac{1}{\text{AREA } D}$$

SO WE TRY

$$\frac{1}{8}\int_{-1}^{1}\left(\int_{-2}^{2} 2e^{-|x|-|y|} - 3 \, dy\right) dx$$

AREA OF RECT.

$$\| $$

$$\frac{1}{8}\int_{-1}^{1}\left(2\int_{0}^{2} 2e^{-|x|-|y|} - 3 \, dy\right) dx$$

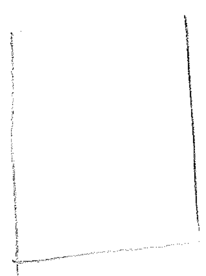
$$= \frac{1}{8}\int_{-1}^{1} \left[-4e^{-|x|-|y|} - 6y\right]_{y=0}^{y=2} dx =$$

$$= \frac{1}{8}\int_{-1}^{1} -4\left(e^{-|x|}(e^{-2}-1)\right) - 12 \, dx =$$

$$\frac{1}{2}\int_{0}^{1} -2e^{-x}(e^{-2}-1) - 6 \, dx = 4\left[2e^{-x}(e^{-2}-1) - 6x\right]_{0}^{1}$$

$$= \frac{1}{2}\left(2(e^{-2}-1)(e^{-1}-1) - 6\right) \approx -2.453$$

OPPOSITE ORDER:



$2 \geq y \geq -2$

$$\int_{-2}^{2}\left(\int_{-1}^{1} 2e^{-|x|-|y|} - 3 \, dx\right) dy$$

$-1 \leq x \leq 1$

ON A RECTANGLE X, Y ARE NOT DEPENDENT ON EACH OTHER SO WE CAN "JUST SWITCH"

$$\int_{c}^{d}\left(\int_{a}^{b} f(x,y)\, dy\right) dx = \int_{a}^{b}\left(\int_{c}^{d} f(x,y)\, dx\right) dy$$

E.G. i) EVALUATE $\int_0^1 \left(\int_0^{\frac{x}{2}} x y^2 \, dy \right) dx$

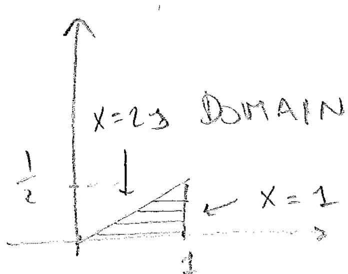
ii) REWRITE $\int_0^1 \left(\int_0^{\frac{x}{2}} x y^2 \, dy \right) dx$ AS

$\int_c^d \left(\int_{a(y)}^{b(y)} x y^2 \, dx \right) dy$ AND EVALUATE IT.

i) $\int_0^1 \left(\int_0^{\frac{x}{2}} x y^2 \, dy \right) dx = \int_0^1 \left[\frac{x y^3}{3} \right]_0^{\frac{x}{2}} dx =$

$$\int_0^1 \frac{x^4}{24} \, dx = \left[\frac{x^5}{120} \right]_0^1 = \frac{1}{120}$$

ii)



$y = \frac{x}{2}$ $\quad x = 2y$

$2y \leq x \leq 1 \quad 0 \leq y \leq \frac{1}{2}$

$$\int_0^{\frac{1}{2}} \int_{2y}^1 x y^2 \, dx \, dy = \int_0^{\frac{1}{2}} \left[\frac{x^2 y^2}{2} \right]_{2y}^1 dy =$$

$$\int_0^{\frac{1}{2}} \frac{y^2}{2} - 2y^4 \, dy = \left[\frac{y^3}{6} - \frac{2}{5} y^5 \right]_0^{\frac{1}{2}} =$$

$$\frac{1}{48} - \frac{2}{160} = \frac{5}{240} - \frac{3}{240} = \frac{1}{120}$$