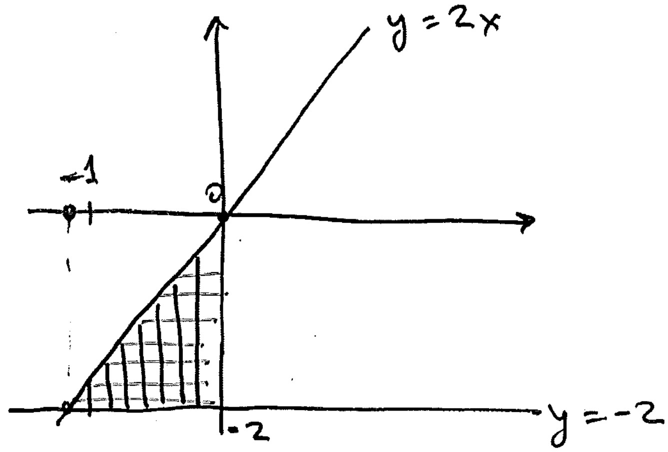
Today: Integrals over regions.

1) Interchanging the order of integration.

Example: $\int_{-1}^{0} \int_{-2}^{2x} e^{y^2}\, dy\, dx$ - evaluate this integral.

Catch: e^{y^2}: its antiderivative does not have a formula in terms of powers, exponentials, ... (elementary functions).

Try changing the order!



Answer:

$$\int_{-2}^{0} \int_{y/2}^{0} e^{y^2}\, dx\, dy$$

recall: $y = 2x$
solve for x:
$x = \frac{y}{2}$
(want x to be a function of y)

$$= \int_{-2}^{0} \int_{y/2}^{0} e^{y^2}\,dx\,dy = \int_{-2}^{0} e^{y^2}\cdot\left(0-\frac{y}{2}\right) dy$$

$$= -\frac{1}{2}\int_{-2}^{0} y\cdot e^{y^2}\,dy = -\frac{1}{2}\int_{4}^{0} \cdot\frac{1}{2}\, e^{u}\,du$$

↑ this helps! Now we can do substitution: $u = y^2$

$du = 2y\,dy$

$$= +\frac{1}{4}\int_{0}^{4} e^{u}\,du = \frac{1}{4}\left(e^{4}-1\right)$$

— — — —

Note: variation of this example: (improper integral):

$\int_{-1}^{0}\int_{-2}^{2x} \frac{e^y}{y}\,dy\,dx$; looks like the same problem, $\frac{e^y}{y}$ cannot be integrated in elementary functions,

if we switch the order of integration, get extra "y", all seems fine.

But this solution would be incorrect!

But this integral is improper ($\frac{e^y}{y} \to -\infty$ at the tip  of our domain)

integral doesn't converge

you cannot change the order!

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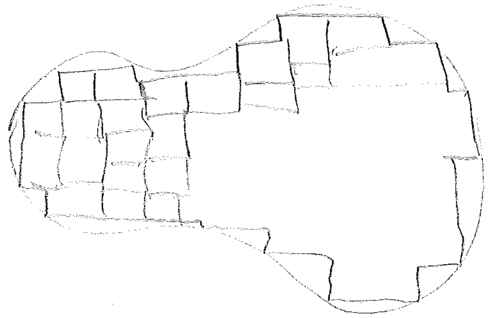
STEP 3:

GENERAL DOMAIN



WE APPROX THE DOMAIN WITH AN INCREASING NUMBER OF RECTANGLES

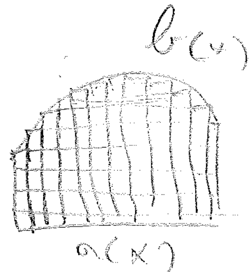
$r_1, \dots, r_m$. PICK x_i^*, y_i^* ON r_i,

THEN $\displaystyle\iint_R f(x,y)\, dA = $

$$\lim_{m\to\infty} \sum_{i=1}^{m} f(x_i^*, y_i^*)\, \text{AREA}\, r_i$$

IF R IS $c \le x \le d$, $a(x) \le y \le b(x)$

THEN BY PICKING A GOOD APPROX

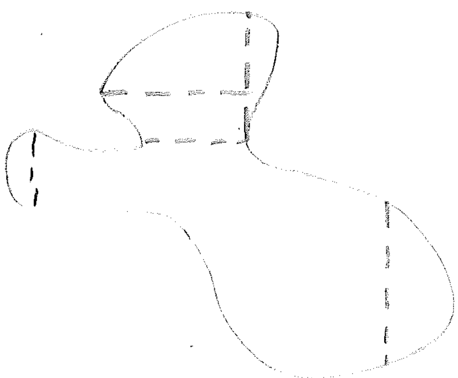


AND REDISTRIBUTING SUMS AS BEFORE WE GET

$$\iint_R f(x,y)\, dA = \int_c^d \int_{a(x)}^{b(x)} f(x,y)\, dy\, dx$$

(SAME FOR $dx\,dy$)

IN PRACTICE TO INTEGRATE ON A COMPLICATED DOMAIN WE CUT IT UP INTO PIECIES WHICH CAN BE DESCRIBED AS THE REGIAN BETWEEN TWO CURVES;



THIS THEORETICAL POWER GIVES US SOME IMMEDIATE PROPERTIES

1) IF $R_1 \cap R_2 = \emptyset$ THEN

$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_1} f(x,y)\,dA$$

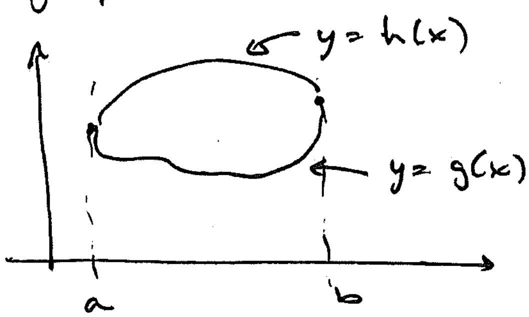
2) IF $f(x,y) = h(x)g(y)$ THEN

$$\int_c^d \int_a^b f(x,y)\,dy\,dx = \int_c^d h(x)\,dx \int_a^b g(y)\,dy$$

3) "A SYMMETRY OF $f(x,y)$ AND R GIVES A SYMMETRY OF $\iint_R f(x,y)\,dA$!!

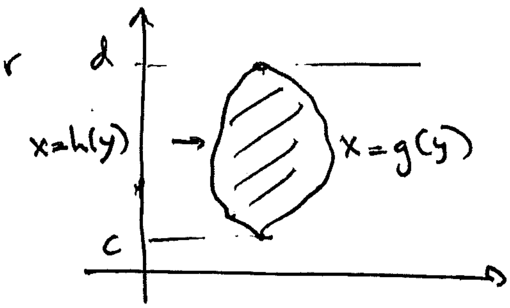
② Polar coordinates

- So far, we can deal with domains bounded by graphs of functions:



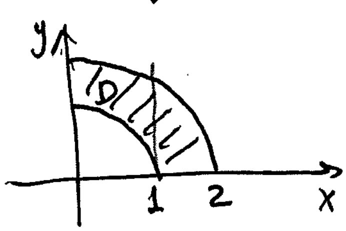
$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)\, dy\, dx$$

or



$$\int_c^d \int_{h(y)}^{g(y)} f(x,y)\, dx\, dy.$$

- what if the domain is bounded by arcs of circles? It might be inconvenient to deal with the functions:

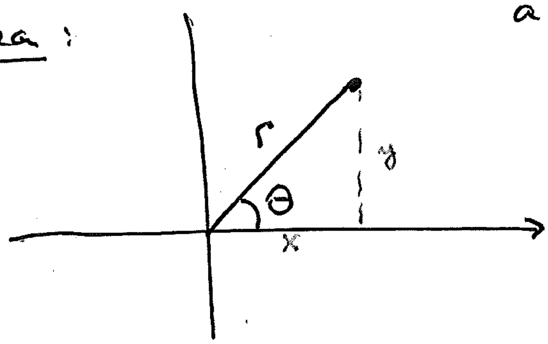


$$\iint_D f(x,y)\, dA$$

- can be done in <u>cartesian</u> (xy-) coordinates, but complicated.

Polar coordinates (Read 9.4) and 13.3

idea:



a point on the plane can be specified by:

- its (x,y) coords.
- distance from $(0,0)$ and direction:

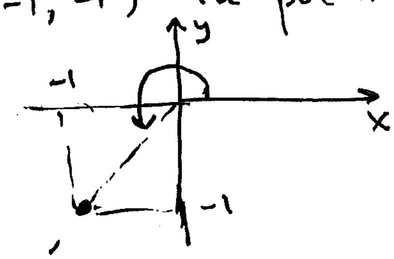
θ - angle from the positive x-axis

r = distance from $(0,0)$

$0 \leq \theta < 2\pi$

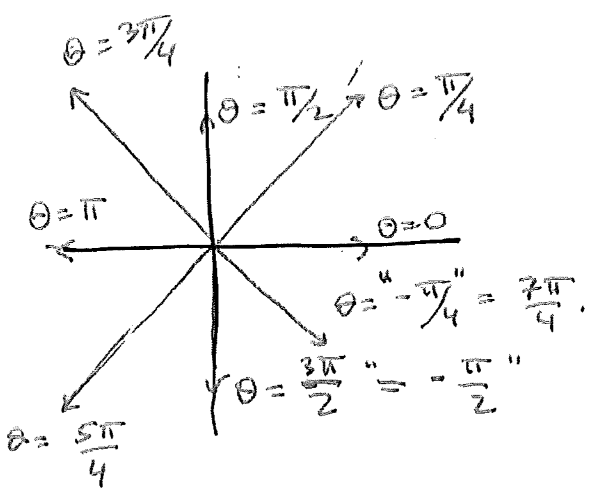
$r \geq 0$.

Example: $(-1,-1)$ in polar coordinates has expression:



$\theta = \frac{5\pi}{4}$,

$r = \sqrt{2}$



Conversion formulas

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \end{aligned}}$$

$\theta = \arctan \frac{y}{x}$ ← if you are in the upper half-plane

$= \pi + \arctan \frac{y}{x}$

in the lower half-plane.

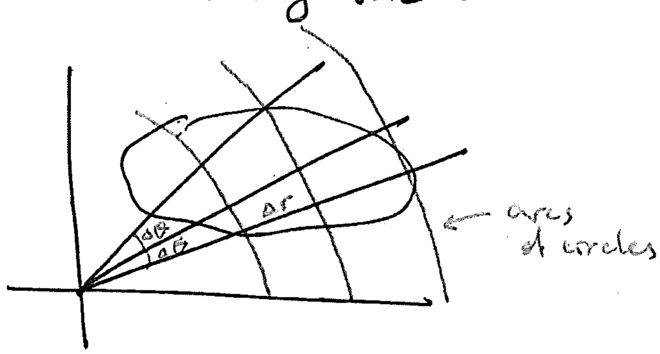
Area in polar coordinates

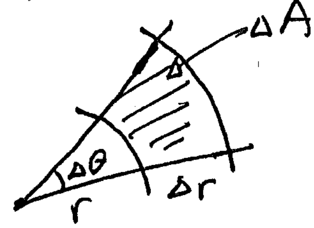
(want to change (x,y) to polar: (r,θ) in our double integrals).

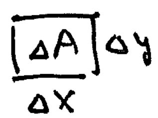
For this, need to express "dA" in polar coords:

recall: the def. of integral was based on cutting the domain into small rectangles.

Now we want to cut into "wedges":







$\Delta A = \Delta x \Delta y$
$= \Delta y \Delta x$

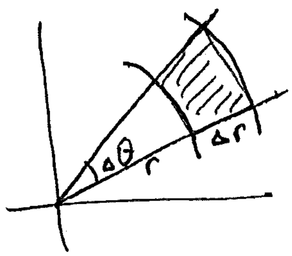
Need to express ΔA
- in terms of r, Δr, $\Delta\theta$.

Turns out: $\Delta A \approx r \cdot \Delta r \cdot \Delta\theta$

Today: • integration in polar coordinates

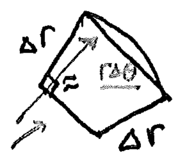
• Mass, centre of mass.

Recall last time:



$\Delta A \approx r\,\Delta r\,\Delta\theta$

Why this works: approximate arcs of circles by straight lines:



pretend the angles are straight.

pretending they are equal.

both approximately = length of the arc of a circle of angular measure $\Delta\theta$, and radius r.

$$\boxed{\Delta A \approx (r\Delta\theta)\cdot\Delta r}$$

recap about π and lengths of arcs:

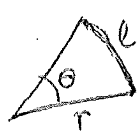
$\theta = 2\pi \longleftrightarrow$ length $= 2\pi r$

$\theta = \pi \longleftrightarrow$ half a circle, length $= \pi r$

$\theta = \frac{\pi}{2} \longleftrightarrow$ quarter of a circle, length $= \frac{\pi}{2} r$

$\theta \longleftrightarrow$ length $\ell = \theta \cdot r$

in radians!

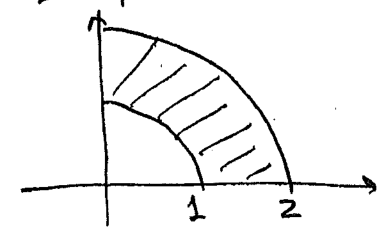


1 radian – angular measure of the angle giving arc length $= 1$ on the circle of radius 1).

The point: $\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,\underline{\underline{r}}\,dr\,d\theta$

plug in $x = r\cos\theta$
$y = r\sin\theta$

Example



came from $\underline{\underline{dA}}$.

Find $\iint_D (x^2+y^2)\,dA = \int_1^2 \int_0^{\pi/2} r^2 \cdot \underline{\underline{r}}\,d\theta\,dr$

$= \frac{\pi}{2} \cdot \int_1^2 r^3\,dr = \frac{\pi}{2} \cdot \frac{r^4}{4}\Big|_1^2$

$= \frac{\pi}{2} \cdot \frac{1}{4} \cdot 15$

D: $1 \le r \le 2$
$0 \le \theta \le \pi/2$

recall: $r = \sqrt{x^2+y^2}$
$x^2+y^2 = r^2 \quad (= r^2\cos^2\theta + r^2\sin^2\theta)$

Note: earlier, when writing iterated integrals in xy-coordi,
- we discussed: constant limits $\Leftrightarrow$ domain is a rectangle.

- in polar coordinates, both limits constant $\Leftrightarrow$
  - between two circles
  - between two rays.