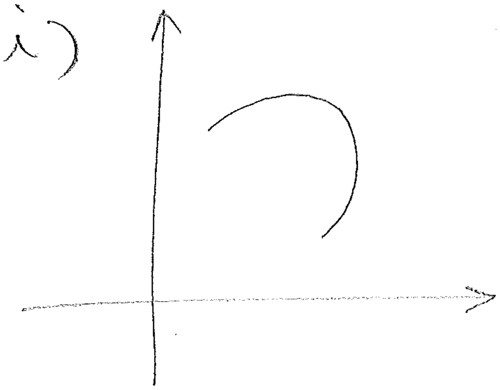
WORKSHEET:

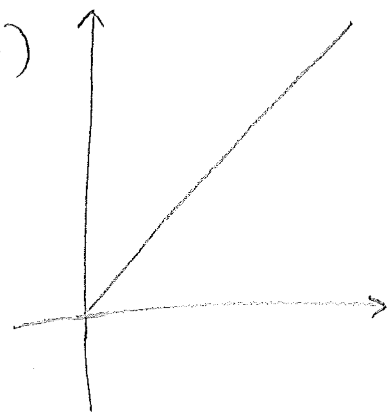
1) WHAT ARE THE "VERTICAL" AND "HORIZONTAL" LINE TESTS IN POLAR COORDINATES?

FOR EACH OF THE FOLLOWING WRITE θ IF THE CURVE IS A FUNCTION OF θ ($r = a(\theta)$), r IF $\theta = a(r)$, N IF NEITHER, B IF BOTH

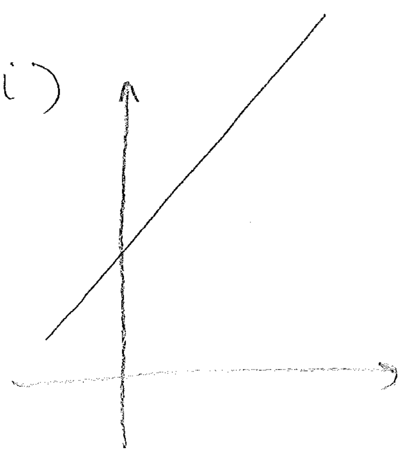
i)



ii)



iii)



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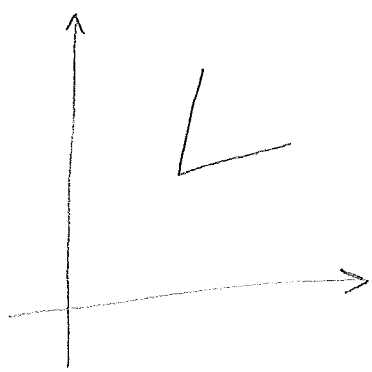

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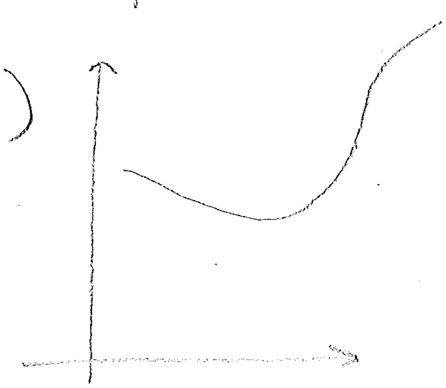
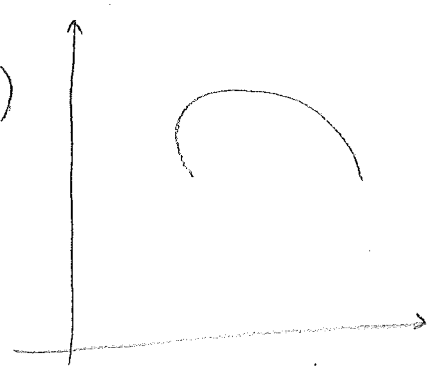
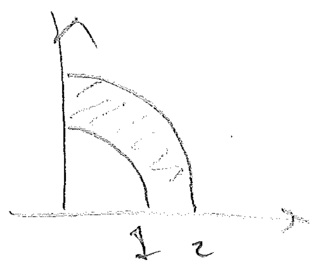


vi)



2) FIND

$$\iint_D (x^2+y^2)\,dA \quad \text{WHERE}$$

$D =$ 

USING POLAR COORDINATE

3) WRITE

$$\iint_D \frac{\sqrt{xy}}{\sqrt{x^2+y^2}}\,dA \quad \text{WHERE } D =$$

$(x-1)^2+(y-1)^2 \leq 1$ AS A 1-VAR INT

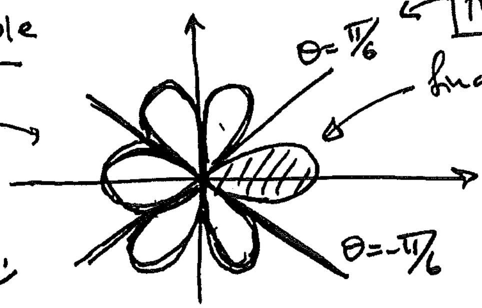
USING POLAR COORDINATES.

Application : mass and centre of mass. (next time).

First: total area : how do you find area of a region?

Example

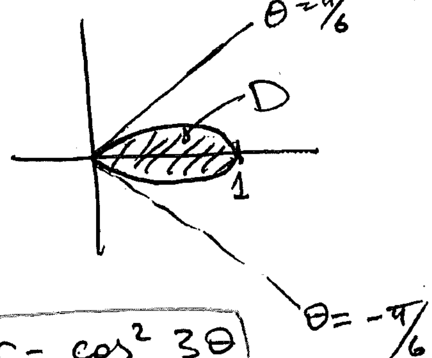
all the petals should be equal ☺



← these lines are tangent to the petals

find the area.

(see 'gallery of polar curves' p. 533 in the book)



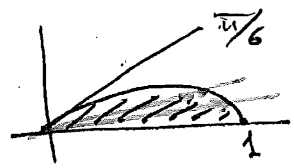
$\boxed{r=\cos^2 3\theta}$

- equation of the flower. (see 9.4).

① $\boxed{\text{Area} = \iint_D 1\, dA.}$

② Set up the integral in polar coordinates.

how to set up limits of integration?

1. 

$0 \le \theta \le \pi/6$

compute by:

$r=0$

$\cos^2 3\theta = 0$

$3\theta = \frac{\pi}{2} + \pi k$

$\theta = \frac{\pi}{6} + \frac{\pi}{3}k$

2. Given: $r=\cos^2 3\theta$ is the boundary.

So: $0 \le r \le \cos^2 3\theta$

$$A = 2\int_0^{\pi/6}\int_0^{\cos^2 3\theta} 1 \cdot r\,dr\,d\theta$$

$$\left(\text{also: } A = \int_{-\pi/6}^{\pi/6}\int_0^{\cos^2 3\theta} 1 \cdot r\,dr\,d\theta\right)$$

$$= 2\int_0^{\pi/6} \frac{r^2}{2}\Big|_0^{\cos^2 3\theta} d\theta = \int_0^{\pi/6} \cos^4(3\theta)\,d\theta$$

$$\left[\ \underline{\text{Recall}}: \quad \cos^2 x = \frac{1+\cos(2x)}{2}\right.$$

$$\cos^4(3\theta) = \left(\cos^2(3\theta)\right)^2 = \left(\frac{1+\cos(6\theta)}{2}\right)^2$$

$$= \frac{1}{4} + \frac{1}{2}\cos(6\theta) + \cos(6\theta)^2 \cdot \frac{1}{4}$$

$$\left. = \frac{1}{4} + \frac{1}{2}\cos(6\theta) + \frac{1}{4}\cdot\frac{1}{2}\left(1+\cos(12\theta)\right)\ \right]$$

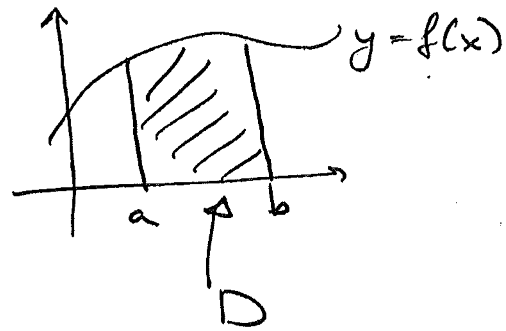
Then $\displaystyle\int_0^{\pi/6} \cos^4(3\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/6}\left(1 + 2\cos(6\theta) + \frac{1}{2}\cos(12\theta)\right)d\theta$

$$= \frac{1}{4}\cdot\left(\frac{\pi}{6} + \underbrace{2\sin(6\theta)\cdot\frac{1}{6}\Big|_0^{\pi/6}}_{=0} + \frac{1}{2}\cdot\underbrace{\sin(12\theta)\cdot\frac{1}{12}\Big|_0^{\pi/6}}_{=0}\right)$$

$$= \boxed{\frac{\pi}{24}}.$$

<u>Note</u>: how this way of finding area relates to calculus-101:

there we had:



area under the graph of $y = f(x)$

$= \int_a^b f(x)\,dx$.

What if we use today's method to compute this area? Let D be the shaded domain under the graph (see picture).

Then today's formula says:

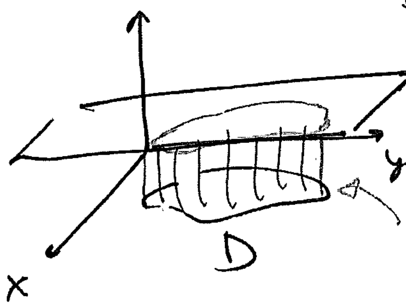
$$\text{area of } D = \iint_D 1\,dA \underset{\substack{\text{encode the}\\ \text{domain } D\\ \text{into the limits}\\ \text{of integration}}}{=} \int_a^b \int_0^{f(x)} 1\,dy\,dx$$

$$\underset{\substack{\text{evaluate}\\ \text{the inner}\\ \text{integral}}}{=} \int_a^b f(x)\,dx$$ — of course, agrees with Math 101.

The point of today's formula is that now we can find areas of more general domains than the domain under the graph of a positive function. (such as the flower from the example)

Recall: Area of $D = \iint_D 1 \, dA$

Why:



$z=1$ – graph of the function $f(x,y)=1$.

volume over D under the graph $= \iint_D 1 \, dA$

$= \begin{pmatrix}\text{base} \\ \text{area}\end{pmatrix} \cdot \text{height}$

$\begin{pmatrix}\text{base} \\ \text{area}\end{pmatrix} = \begin{pmatrix}\text{area} \\ \text{of } D\end{pmatrix}$, height $= 1$

Mass and centre of mass

1) Total mass.

imagine a metal plate of variable density:

mathematical model for it:

a domain D with
<u>density function</u> $\rho(x,y)$.

What is <u>density</u>?

– mass/volume – 3^d-density.

<u>Here</u>: $\dfrac{\text{mass}}{\text{area}}$ ← thin plate density (2^d)

$\dfrac{\text{mass}}{\text{length}}$ ← for a <u>wire</u> (1^d).

Refinement: when density is not constant,

$$\frac{\text{total mass}}{\text{total area}} = \text{average density};$$

but to compute density around a point (x_0, y_0),

we take small square around (x_0, y_0), compute

$$\rho(x_0,y_0) = \lim_{\substack{\text{size} \\ \text{of } \square \to 0}} \frac{\text{mass}(\square)}{\text{area}(\square)}$$

So: $\boxed{\text{total mass of the plate} = \iint_D \rho(x,y)\, dA}$

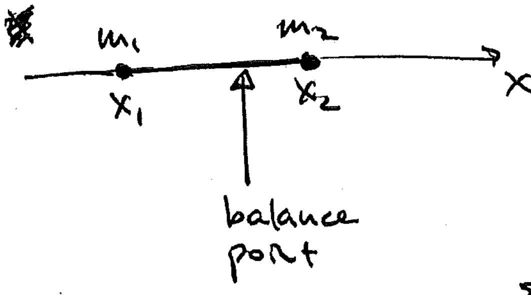
Note: if density $\rho(x,y)$ is constant $= 1 \frac{g}{mm^2}$ ← units of mass / units of area

$$\text{total mass of a lamina} = \iint_D 1 \cdot dA = (\underbrace{\text{area of } D}_{\text{in } mm^2}) \cdot \frac{g}{mm^2}$$

Centre of mass

if place a needle at the centre of mass the plate would be balanced.

Def (for a finite mass system on a line):



$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

centre of mass

weighted average of x_1, x_2

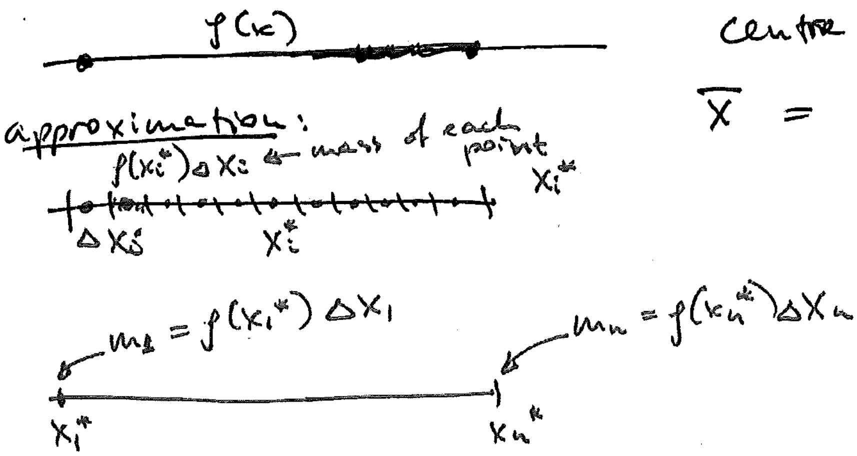
Note: $= \frac{x_1 + x_2}{2}$ = midpoint if $m_1 = m_2$.

(Note: expected value in probability is the same thing).

- For $\underline{\underline{n}}$ points: $x_1, \dots x_n$ – masses $m_1, \dots m_n$

$$\overline{x} = \frac{m_1 x_1 + \dots + m_n x_n}{m_1 + \dots + m_n} \leftarrow \text{total mass.}$$

- Continuous wire of density $\rho(x)$:



centre of mass:

$$\overline{x} = \frac{\int_a^b x\,\rho(x)\,dx}{\text{total mass}}$$

$$\left[\ \text{total mass} = \int_a^b \rho(x)\,dx.\ \right]$$

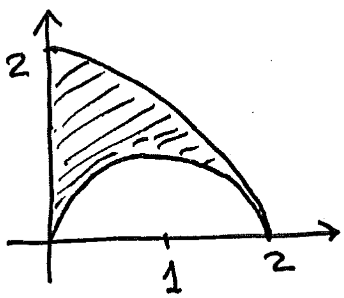
- $\underline{\text{In } 2^d}$: $\underline{\text{Mass}}$ $\quad M = \iint_D \rho(x,y)\,dA$

$\rho(x,y)$ – $\underline{\text{density}}$:

$$\left[\begin{aligned} \overline{x} &= \frac{1}{M}\underbrace{\iint_D x\,\rho(x,y)\,dA}_{} \leftarrow M_y \text{ – moment about } y\text{-axis} \\ \overline{y} &= \frac{1}{M}\underbrace{\iint_D y\,\rho(x,y)\,dA}_{} \leftarrow M_x = \text{moment about } x\text{-axis.} \end{aligned}\right]$$

$$\boxed{\overline{x} = \frac{M_y}{M}, \quad \overline{y} = \frac{M_x}{M}}$$

Worksheet 10



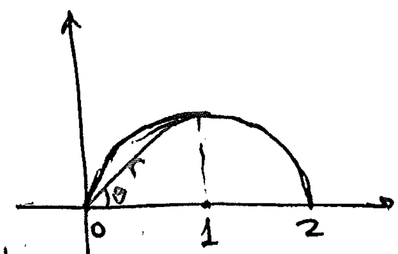
A lamina occupies the area pictured (between the circle of radius 2 centered at (0,0) and the circle of radius 1 centred at (1,0)).

Its density is $\rho(x,y) = \sqrt{x^2+y^2}$.

Find the total mass and the y-coordinate of the centre of mass.

The difficulty:

equation of this circle in polar coordinates



to get it, start with xy:

$$(x-1)^2 + y^2 = 1$$

$$(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$$

$$r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$$

$$r^2 - 2r\cos\theta = 0$$

$$\boxed{r = 2\cos\theta}$$

$$M = \iint_D \rho(x,y)\,dA \underset{\text{in polar}}{=} \int_0^{\pi/2}\int_{2\cos\theta}^{2} \overset{\text{from } \rho}{r} \cdot \overset{\text{from } dA}{r}\,dr\,d\theta$$

$$\rho(x,y) = \sqrt{x^2+y^2} = r$$

$$= \int_0^{\pi/2}\int_{2\cos\theta}^{2} r^2\,dr\,d\theta = \frac{1}{3}\int_0^{\pi/2} r^3\Big|_{2\cos\theta}^{2}\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\left(8 - 8\cos^3\theta\right)d\theta$$

$$= \frac{8}{3}\cdot\frac{\pi}{2} - \frac{8}{3}\int_0^{\pi/2}\overbrace{(1-\sin^2\theta)\cos\theta}^{\cos^3\theta}\,d\theta \overset{u=\sin\theta}{=} \frac{4\pi}{3} - \frac{8}{3}\int_0^1 (1-u^2)\,du = \frac{4\pi}{3} - \frac{8}{3}\left(1-\frac{1}{3}\right) = \boxed{\frac{4\pi}{3} - \frac{16}{9}}$$

this is M.

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} \underset{x}{(r\cos\theta)} \cdot \underset{\rho}{r} \cdot \underbrace{r\,dr\,d\theta}_{dA} \\ \bar{y} &= \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} (r\sin\theta)\cdot r\cdot r\,dr\,d\theta \end{aligned}}$$

Computing $\bar{x}$ and $\bar{y}$: (actually, the question was only about $\bar{y}$, but I'll do both).

$$\bar{x} = \frac{1}{M}\int_0^{\pi/2}\int_{2\cos\theta}^{2} r^3\cos\theta\,dr\,d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2} \frac{r^4}{4}\Big|_{2\cos\theta}^{2}\cdot\cos\theta\,d\theta$$

$$= \frac{1}{M}\int_0^{\pi/2}\left(4 - 4\cos^4\theta\right)\cdot\cos\theta\,d\theta$$

$$= \frac{4}{M}\int_0^{\pi/2}\cos\theta - \cos^5\theta\,d\theta = \frac{4}{M}\cdot\sin\theta\Big|_0^{\pi/2} - \frac{4}{M}\int_0^{\pi/2}\cos^5\theta\,d\theta$$

As before an odd power of $\cos\theta$ is integrated using the change of variable $u = \sin\theta$:

$$\int_0^{\pi/2} \cos^5\theta \, d\theta = \int_0^1 (1-u^2)^2 \, du = \int_0^1 (1 - 2u^2 + u^4) \, du$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$\cos^4\theta = (1-\sin^2\theta)^2$

$$= 1 - 2 \cdot \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

Putting it together, get:

$$\overline{x} = \frac{4}{M} - \frac{4}{M} \cdot \frac{8}{15} = \boxed{\frac{4}{M} \cdot \frac{7}{15}}$$

(M as above)

$M = \frac{4\pi}{3} - \frac{16}{9}$

$$\overline{y} = \frac{1}{M} \int_0^{\pi/2} \int_{2\cos\theta}^{2} \underbrace{r\sin\theta}_{y} \cdot \underbrace{r}_{\rho(x,y)} \cdot r \, dr \, d\theta$$

$$= \frac{1}{M} \int_0^{\pi/2} \left(\int_{2\cos\theta}^{2} r^3 \, dr \right) \sin\theta \, d\theta$$

$$= \frac{1}{M} \int_0^{\pi/2} \left. \frac{r^4}{4} \right|_{2\cos\theta}^{2} \sin\theta \, d\theta$$

$$= \frac{4}{M} \int_0^{\pi/2} (1 - \cos^4\theta) \sin\theta \, d\theta =$$

$\boxed{u = \cos\theta}$

$du = -\sin\theta \, d\theta$

$$= \frac{4}{M} \left(-\int_1^0 (1-u^4) \, du \right) = \frac{4}{M} \int_0^1 (1-u^4) \, du$$

$$= \frac{4}{M} \cdot \left(1 - \frac{1}{5}\right) = \boxed{\frac{16 M}{5}}.$$

$\boxed{M \text{ as above}}$