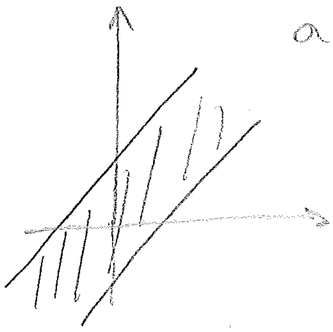
# CRITICAL POINTS, EXTREME VALUES

GOAL FOR THE NEXT FEW CLASSES:

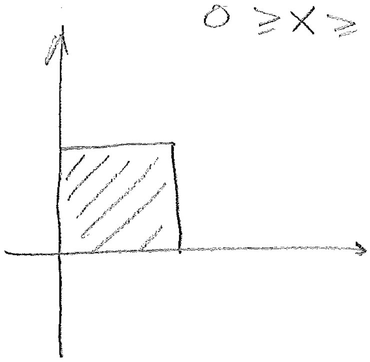
FIND MIN/MAX OF $f(x,y)$ (OR $f(x,y,z)$) ON A REGION $D$.

$D$ CAN BE MANY DIFFERENT REGIONS:

$a \geq x - y \geq b$

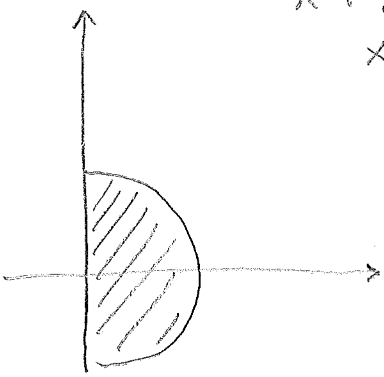


$0 \geq x \geq 1 \quad 0 \geq y \geq 1$

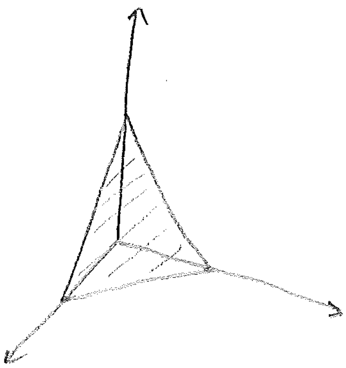


$x^2 + y^2 \leq 4$

$x \geq 0$



$x + y + z \leq 1 \quad x, y, z \geq 0$



TWO SOURCES OF DIFFICULTY:

- THE FUNCTION $f$
- THE REGION $D$

STEP 1: FINDING THE CRITICAL POINTS FOR $f(x,y)$ (OR $f(x,y,z)$).

DEF: P IS A CRITICAL POINT FOR $f$ IF $\vec{\nabla} f \big|_P = \vec{0}$ OR AT LEAST ONE PARTIAL DERIVATIVE IS NOT DEFINED.

(MORE PRECISELY: $f$ IS NOT DIFFERENTIABLE AT P)

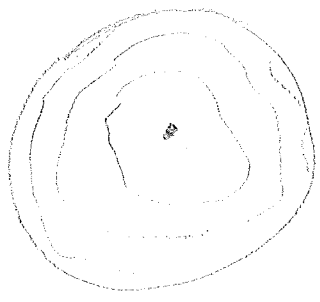
DEF: $(a,b)$ (RESPECTIVELY $(a,b,c)$) IS A LOCAL MAX FOR $f(x,y)$ ($f(x,y,z)$) IF THERE IS A DISK (A SPHERE) AROUND $(a,b)$ ($(a,b,c)$) SUCH THAT FOR ALL POINTS $(a',b')$ IN THE DISK

$$f(a,b) \geq f(a',b')$$

RESPECTIVELY: FOR ALL $(a',b',c')$ IN THE SPHERE

$$f(a,b,c) \geq f(a',b',c').$$

LOCAL MINIMUM: SAME WITH $\leq$.

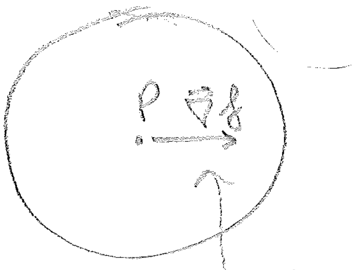


EXAMPLE: GROUSE MTN PEAK IS A LOCAL MAX FOR ALTITUDE AS A FUNCTION OF NS/WE COORDINATE

EXAMPLE: THE CENTER OF THE SUN IS A LOCAL MAX FOR TEMPERATURE IN THE SOLAR SYSTEM.

THEOREM: A LOCAL MIN/MAX CAN ONLY OCCUR AT A CRITICAL POINT.

WHY:




CANNOT HAPPEN

P LOCAL MIN/MAX.

IF $\vec{\nabla} f \neq \vec{0}$ THEN ALONG THE DIRECTION $\vec{u}$ OF $\vec{\nabla} f$ THE FUNCTION INCREASES,

BUT THEN FOR SOME SMALL $h$ WE MUST HAVE $f(P + h\vec{u}) > f(P)$ AND $f(P - h\vec{u}) < f(P)$ SO P CANNOT BE A MAX OR MIN.

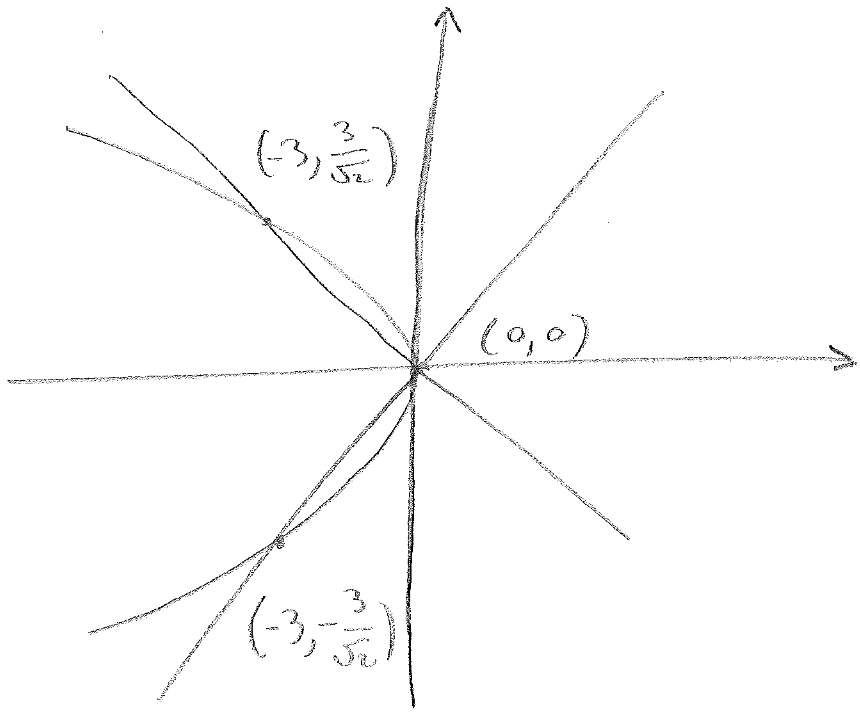
E.G. FIND THE CRITICAL POINTS OF

$$f(x,y) = x^3 + x^2y^2 - y^4$$

$$f_x = 3x^2 + 2xy^2 \qquad f_y = 2yx^2 - 4y^3 =$$

$$= x(3x + 2y^2) \qquad y(2x^2 - 4y^2)$$

$$\begin{cases} x(3x+2y^2) = 0 \Rightarrow x = 0 \text{ OR } x = -\frac{2}{3}y^2 \\ y(2x^2 - 4y^2) = 0 \Rightarrow y = 0 \text{ OR } \sqrt{2}x + 2y = 0 \\ \qquad \text{OR } \sqrt{2}x - 2y = 0 \end{cases}$$




INTERSECTIONS: $X = 0$, $y = 0$

AND
$$\begin{cases} x = -\frac{2}{3} y^2 \\ X = -\sqrt{2}\, y \end{cases} \sim \begin{cases} \frac{2}{3} y^2 = \sqrt{2}\, y \\ X = -\sqrt{2}\, y \end{cases} \sim$$

$$\begin{cases} y \left( \frac{2}{3} y - \sqrt{2} \right) = 0 \\ - \end{cases} \sim \begin{cases} y = \frac{3}{\sqrt{2}} \\ x = -3 \end{cases}$$

$$\begin{cases} x = -\frac{2}{3} y^2 \\ x = \sqrt{2}\, y \end{cases} \sim \ldots \sim \begin{cases} y = \frac{-3}{\sqrt{2}} \\ x = -3 \end{cases}$$

SO CRITICAL POINTS: $(0,0)$, $(-3, \frac{3}{\sqrt{2}})$, $(-3, \frac{-3}{\sqrt{2}})$

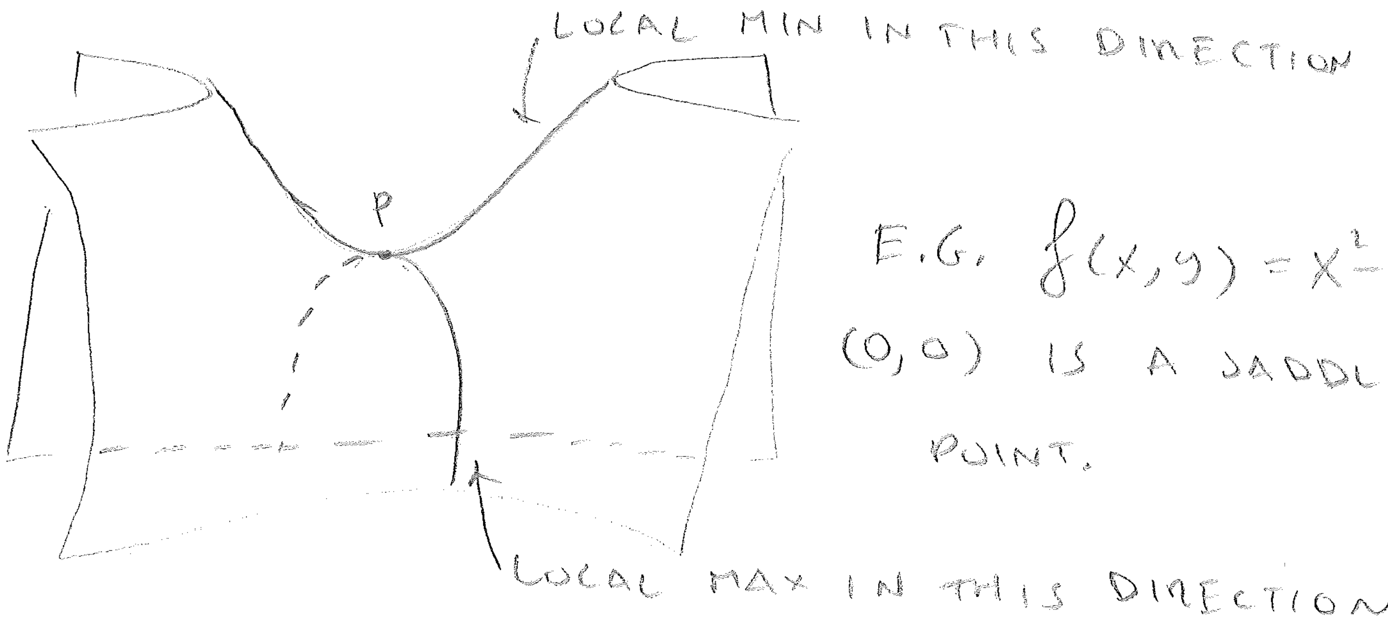
OK, WE FOUND THE CRITICAL POINTS. WHAT NOW? WE LOOK AT HIGHER DERIVATIVES.

A CRITICAL PT CAN BE A:

- LOCAL MAX
- LOCAL MIN
- SADDLE POINT
- NEITHER

SADDLE POINT:

LOCAL MAX ALONG A DIRECTION, LOCAL MIN ALONG ANOTHER




E.G. $f(x,y) = x^2 - y^2$, $(0,0)$ IS A SADDLE POINT.