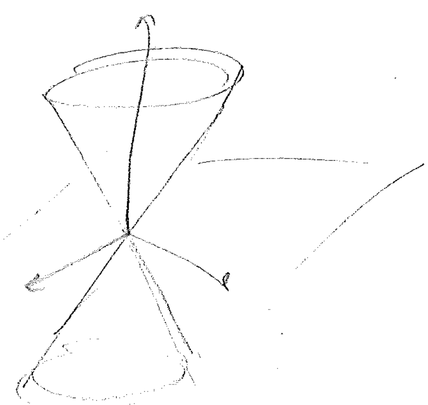
E.G. $f(x,y,z) = x^2 - 3z^2 + y^2$.

i) DESCRIBE THE LEVEL SURFACES OF f.

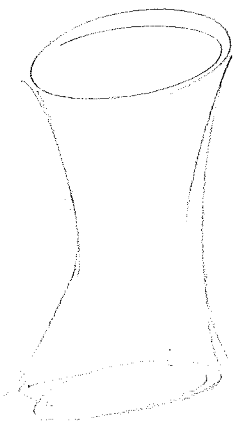
ii) FIND A NORMAL VECTOR TO THE TANGENT PLANE TO $f(x,y,z) = -22$ AT $P = (1, 2, 3)$

i) $x^2 - 3z^2 + y^2 = K$

$K = 0$ $\quad z^2 = \dfrac{x^2 + y^2}{3}$ CONE

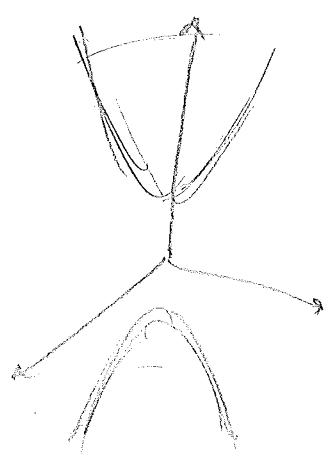


$K > 0$



HYPERBOLOID 1 SHEET

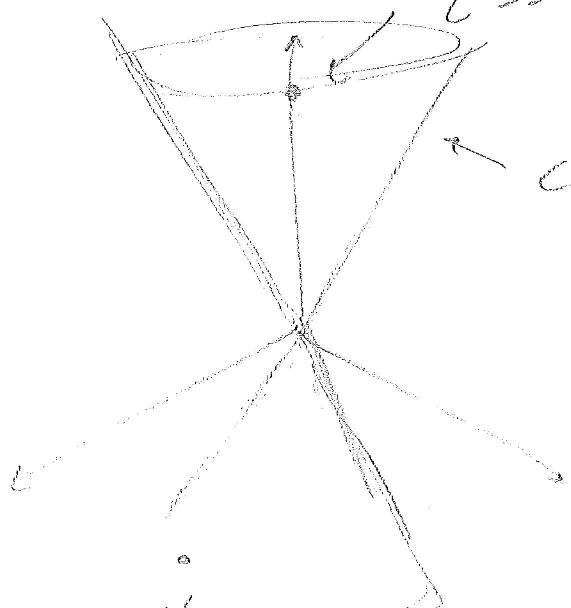
$K < 0$



HYPERBOLOID, 2 SHEETS

ALL TOGETHER? THEY DO NOT TOUCH, SO

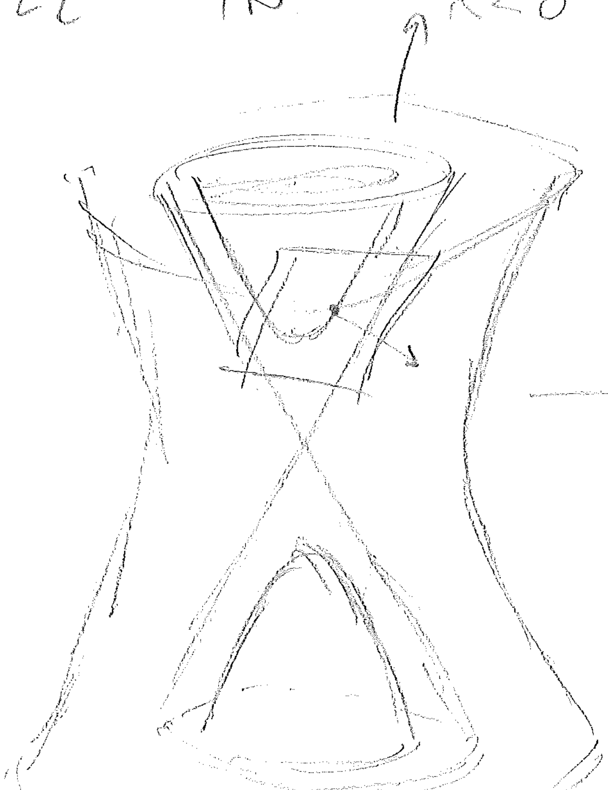
$z >> 0,\ x = y = 0 \quad K < 0$



CONE $K = 0$

x or $y >> 0,\ z = 0 \quad K > 0$

FILL IN $K<0$



$K>0$

NORMAL AT $(1, 2, 3)$

$$\vec{\nabla} g = \langle 2x, 2y, -6z \rangle$$

$$= \langle 2, 4, -18 \rangle$$

TANGENT PLANE

$$2x + 4y - 18z = -44$$

DEF: GIVEN A FUNCTION $f(x,y)$ OR $f(x,y,z)$ THE GRADIENT $\vec{\nabla} f$ IS THE VECTOR

"NABLA" $\langle f_x, f_y \rangle$ (RESPECTIVELY $\langle f_x, f_y, f_z \rangle$).

NOTE: $\vec{\nabla} f$ IS THE NORMAL VECTOR TO THE TANGENT PLANE OF THE LEVEL SURFACES $f(x,y,z) = K$.

WHY:

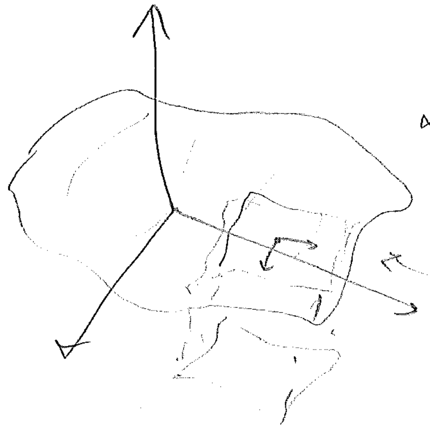
$P = (a,b,c)$; ASSUME $\vec{\nabla} f \neq 0$ AT P (OTHERWISE THE TANGENT PLANE D.N.E.).

UP TO REARRANGING VARIABLES, $f_z \neq 0$.

THEN BY IMPLICIT FUNCTION THM

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}, \quad \frac{\partial z}{\partial y} = \frac{-f_y}{f_z},$$

TANGENT PLANE IS // TO $\langle 1, 0, \frac{-f_x}{f_z} \rangle$ AND $\langle 0, 1, \frac{-f_y}{f_z} \rangle$.



SLOPES OF TANGENTS ARE $\frac{\partial z}{\partial x}$ AND $\frac{\partial z}{\partial y}$

NOW NOTE THAT

$$\vec{\nabla} f \langle 1, 0, \frac{-f_x}{f_z} \rangle = f_x - f_x \frac{f_z}{f_z} = 0$$

$$\vec{\nabla} f \langle 0, 1, \frac{-f_y}{f_z} \rangle = f_y - f_y \frac{f_z}{f_z} = 0$$

SO $\vec{\nabla} f(a,b,c)$ IS $\perp$ TO TANGENT PLANE AT (a,b,c).

ANOTHER WAY TO FIND THE TANGENT PLANE TO THE GRAPH OF $f(x,y)$:

GRAPH OF $f(x,y) = \{z - f(x,y) = 0\}$

SO NORMAL DIRECTION IS

$$\vec{\nabla}\left(z - f(x,y)\right) = \langle -f_x, -f_y, 1 \rangle.$$

DIRECTIONAL DERIVATIVES

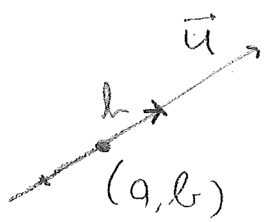
$f = f(x,y)$ OR $f(x,y,z)$

$\vec{u}$ UNIT VECTOR $\vec{u} = \langle u_1, u_2 \rangle$ OR

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

DEF:

2 VAR: $$D_{\vec{u}} f \Big|_{(a,b)} = \lim_{h \to 0} \frac{f(a+u_1 h, b+u_2 h) - f(a,b)}{h}$$



LIMIT APPROACHING FROM THE DIRECTION OF $\vec{u}$

$$= \lim_{h \to 0} \frac{f((a,b) + h\vec{u}) - f(a,b)}{h}$$

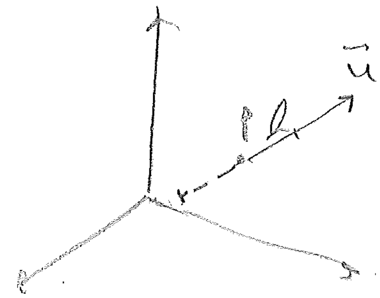
(NOTE: POINT + VECTOR = POINT)

WE ARE EVALUATING THE RATE OF CHANGE OF f IN THE DIRECTION OF $\vec{u}$

3 VAR: $$D_{\vec{u}} f \Big|_{(a,b,c)} = \lim_{h \to 0} \frac{f((a,b,c) + h\vec{u}) - f(a,b,c)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$$

SAME IDEA.



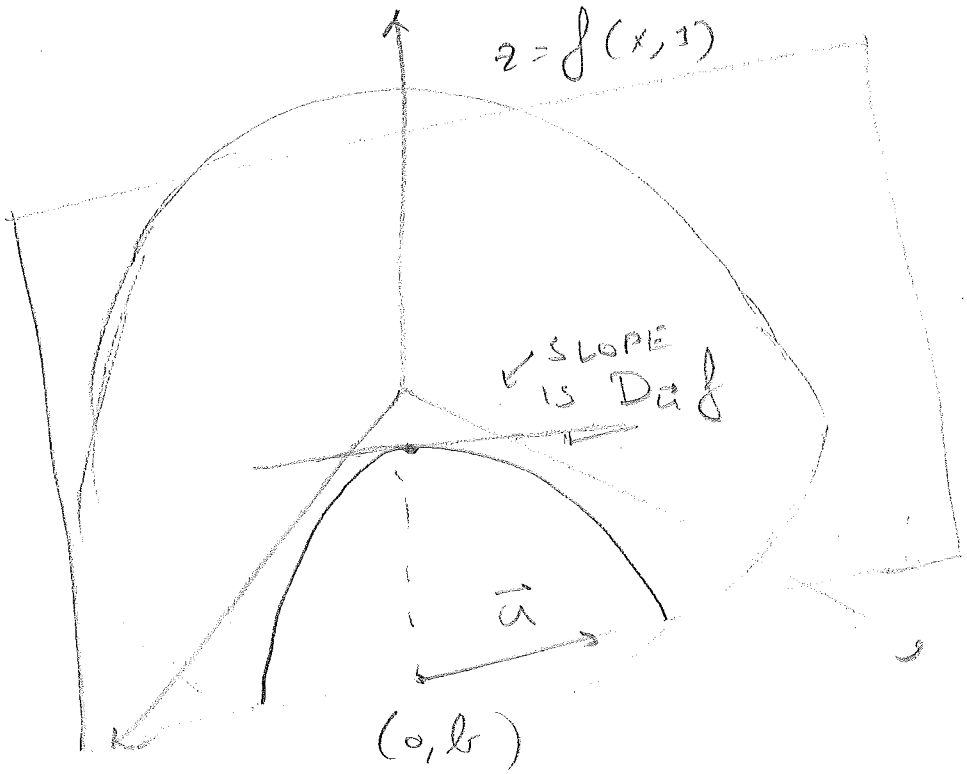
SO $D_{\langle 1,0 \rangle} f \Big|_{(a,b)} = \frac{\partial f}{\partial x} \Big|_{(a,b)}$

$D_{\langle 0,1 \rangle} f = f_y \, , \ldots , \, D_{\langle 0,0,1 \rangle} f = f_z$

3 VAR

<u>GEOMETRICALLY</u> (2-VAR):

$D_{\bar{u}} f \Big|_{(a,b)} =$ SLOPE OF TANGENT LINE AFTER WE INTERSECT THE GRAPH WITH PLANE THROUGH $(a,b,0)$ PARALLEL TO $\bar{u}$ AND z-AXIS.



EXAMPLE:
HALF SPHERE
$f(x,y) = \sqrt{R^2 - x^2 - y^2}$

HOW TO COMPUTE $D_{\vec{u}} f$?

THM:
IF f_x, f_y (AND f_z IN 3 VAR) ARE CONTINUOUS THEN

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

WHY:
CHAIN RULE WITH

$$\sigma(h) = (a + u_1 h,\ b + u_2 h)$$

STRAIGHT PATH PARALLEL TO $\vec{u}$ STARTING AT h.

E.G.

$f(x,y,z) = e^{xy} + 3zx$

$\vec{v} = \langle 1,2,3 \rangle$

FIND THE DIRECTIONAL DERIVATIVE OF f IN THE DIRECTION OF $\vec{v}$ AT $P = (1,5,6)$

1) TURN $\vec{v}$ INTO UNIT VECTOR

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1,2,3 \rangle}{\sqrt{14}} = \left\langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$$

2) $\nabla f \big|_{(1,5,6)} = \ldots$

$f_x = ye^{xy} + 3z \quad f_y = xe^{xy}$

$f_z = 3x$

AT $1,5,6$ $\quad f_x = 5e^5 + 18,$

$f_y = e^5, \quad f_z = 3$

$= \langle 5e^5 + 18, e^5, 3 \rangle$

3)

$$D_{\vec{u}} f = \vec{u} \cdot \nabla f = \frac{1}{\sqrt{14}} (5e^5 + 18 + 2e^5 + 9)$$

$$= \frac{1}{\sqrt{14}} (7e^5 + 27)$$

NEW NOTATION:

GIVEN $\vec{v} \neq 0$, WE DEFINE

$$D_{\vec{v}} f = D_{\frac{\vec{v}}{\|\vec{v}\|}} f .$$

GEOMETRIC MEANING OF $\vec{\nabla}$

IN ANY # OF DIMENSIONS

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

IF f IS "NICE ENOUGH".

- WE CAN DO THIS AT ANY POINT
- MEASURES RATE OF CHANGE ALONG DIRECTION OF $\vec{u}$
- CAREFUL: WE CANNOT TALK OF "SLOPES" ON MORE THAN 2 VAR; FOR $f(x,y,z)$ RATE OF CHANGE MAKES SENSE, SLOPE DOES NOT.

WHAT ARE THE DIRECTIONS OF MAXIMUM AND MINIMUM CHANGE?

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = \|\vec{\nabla} f\| \cdot \cos\theta$$

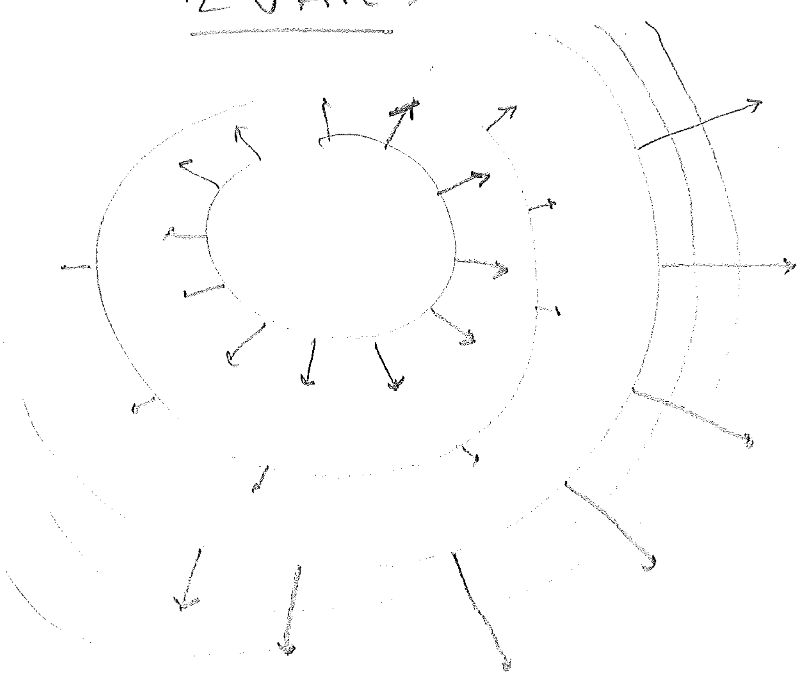
θ = ANGLE BETWEEN $\vec{\nabla} f$ AND $\vec{u}$

SO <u>MAXIMUM INCREASE</u> IS WHEN <u>$\vec{u} \parallel \vec{\nabla} f$, POINTS IN THE SAME DIRECTION</u>.

<u>MAXIMUM</u> DECREASE IS WHEN $\vec{u}$ IS <u>OPPOSITE</u> TO $\vec{\nabla} f$

NO CHANGE WHEN $\vec{u} \perp \vec{\nabla} f$

<u>2 VAR:</u>



CONTOUR DIAGRAM OF $f(x,y)$

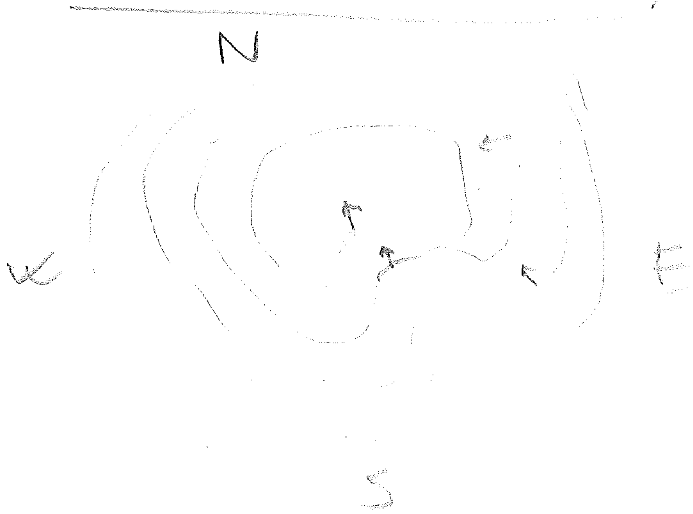
GRADIENT IS LONGER WHEN GRAPH IS STEEPER

$\longleftrightarrow$

LEVEL CURVES ARE CLOSER

REMEMBER: THE GRADIENT LIVES ON THE DOMAIN, NOT ON THE GRAPH!

IN OTHER WORDS:



THE GRADIENT LIVES ON THE MAP, NOT THE MOUNTAIN.

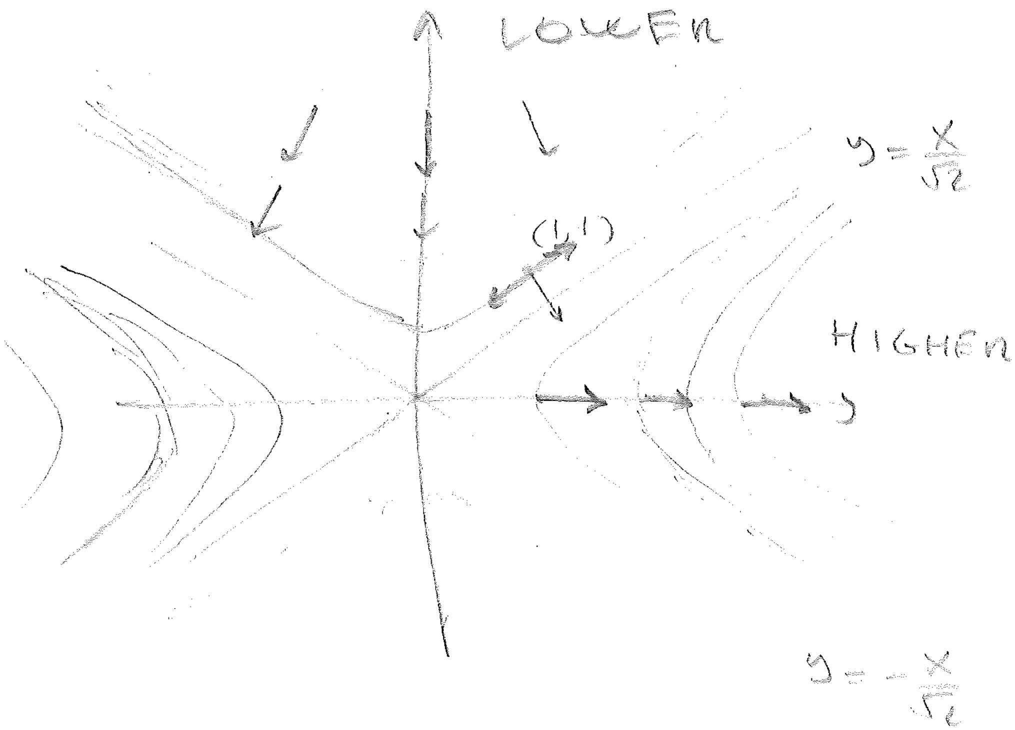
E.G. A HIKER IS CLIMBING UP A MOUNTAIN PASS DESCRIBED, IN TENS OF METERS, BY THE EQUATION $x^2 - 2y^2 = z + 100$. HE IS AT THE POINT $(1, 1)$. WHICH DIRECTION IS STEEPEST? ALONG WHICH DIRECTION THE HEIGHT IS CONSTANT?

$\vec{u}$ DIRECTION

$$D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f = \vec{u} \cdot \langle 2, -4 \rangle$$

MAXIMUM STEEPNESS $\vec{u} = \dfrac{\langle 2, -4 \rangle}{\| \langle 2, -4 \rangle \|} = \left\langle \dfrac{\sqrt{2}}{\sqrt{5}}, \dfrac{-2\sqrt{2}}{\sqrt{5}} \right\rangle$

SAME HEIGHT $\vec{u}^{\perp} = \pm \left\langle \dfrac{2\sqrt{2}}{\sqrt{5}}, \dfrac{\sqrt{2}}{\sqrt{5}} \right\rangle$



E.G. $f(x,y) = -x^2 + 2x - y^2 + 2y + 1$

FIND THE DIRECTIONS OF MAX/MIN INCREASE AT $P = (0,0)$