WORK SHEET:

L.W. IS ON ITS LAWN CHAIR, HELD IN THE AIR BY 25 BALLOONS.

HIS POSITION, IN EAST-WEST, NORTH-SOUTH, UP-DOWN COORDINATE IS $(1,1,0) \cdot 1000\ m$.

THE AIR TEMPERATURE, IN THE SAME COORDINATES CAN BE LOCALLY DESCRIBED AS

$$T(x,y,z) = -20 + 20e^{-y-z+1} + \sqrt{x} \quad C^\circ$$

RIGHT NOW L.W. IS MOVING AT A VELOCITY OF

$\langle 10, 0.25, 0 \rangle\ m/min$.

i) SHOW THAT THE TEMPERATURE AROUND L.W. IS STATIONARY

SOL: $\vec{\nabla} T = \left\langle \frac{1}{2\sqrt{x}}, -20e^{-y-z+1}, -20e^{-y-z+1} \right\rangle$

$$\vec{\nabla} T \Big|_{(1,1,0)} = \left\langle \frac{1}{2}, -20, -20 \right\rangle \frac{C^\circ}{1000\ m}$$

$$\frac{d}{d\tau} T \Big|_{T_0} = \vec{\nabla} T \Big|_{(1,1,0)} \cdot \vec{v} =$$

$$\left\langle \frac{1}{2}, -20, -20 \right\rangle \cdot \left\langle 10, \frac{1}{4}, 0 \right\rangle \frac{C^\circ}{10^3\,\mathrm{min}}$$

$$= 10 - 10 = 0$$

ii) A SUDDEN WIND CHANGES L.W.'S VELOCITY TO $\vec{v}_1 = \langle 10, 1.25, 1.5 \rangle$. L.W. CAN POP BALLOONS WITH HIS BB GUN, AND EACH OF THEM DECREASES THE Z COMPONENT OF HIS VELOCITY BY 0.2. HOW MANY BALLOONS SHOULD HE POP TO MOVE IN THE MOST STATIONARY DIRECTION FOR T?

SOL: $v_2 = \langle 10, 1.25, 1.5 - \frac{b}{5} \rangle$

b = BALLOONS POPPED

WE WANT TO MAKE

$D_{\vec{v}_2} T$ AS CLOSE TO 0 AS POSSIBLE

$$D_{\vec{v}_2} T = \vec{\nabla} T \cdot \frac{\vec{v}_2}{\|\vec{v}_2\|} =$$

$$\frac{1}{\|\vec{v}_2\|} \left\langle \frac{1}{2}, -20, -20 \right\rangle \cdot \left\langle 10, 1.25, 1.5 - \frac{b}{5} \right\rangle =$$

$$\frac{5 - 25 - 30 + 4b}{\|\vec{v}_2\|} = \frac{-50 + 4b}{\sqrt{101.6 + \left(1.5 - \frac{b}{5}\right)^2}}$$

$D_{\vec{v}_2} T = 0$ WHEN $b = 12.5$

SO CANDIDATES 12, 13

$$12: \frac{-2}{\sqrt{101.6 + (0.9)^2}} \qquad 1.1 > 1.1 \qquad 13: \frac{+2}{\sqrt{101.6 + (1.1)^2}}$$

SO BEST CHOICE = 13

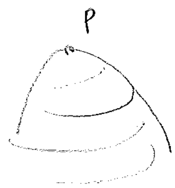
THE SECOND DERIVATIVE TEST

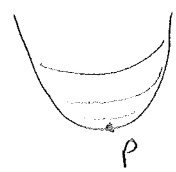
$f(x,y)$, P CRITICAL POINT.

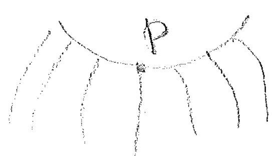
P CAN BE

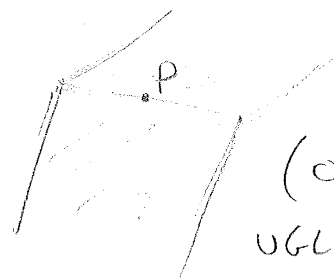
i) LOCAL MAX ii) LOCAL MIN

iii) SADDLE POINT iv) NEITHER / UNDEFINED

i) 

ii) 

iii) 

iv) 

(ONE OF MANY UGLY THINGS)

WHY:

1 VAR:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

2 VAR:

$$f(x,y) \approx f(a,b) + f_x(x-a) + f_y(y-b) +$$

QUADRATIC TERM IN $(x-a)$, $(y-b)$

AT A CRITICAL POINT:

$f(x,y) \approx f(a,b) +$ 1ST ORDER TERM VANISHES + 2ND ORDER TERM

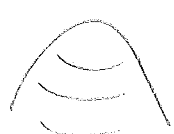
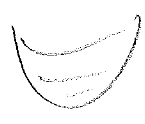
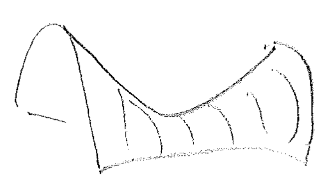
$= f(a,b) + c_1(x-a)^2 + c_2(y-b)^2 + c_3(x-a)(y-b)$

FOR SIMPLICITY, $a = b = f(a,b) = 0$

$f(x,y) = c_1 x^2 + c_2 y^2 + c_3 xy$

YOU CAN ALWAYS COMPLETE THE SQUARE TO "GET RID" OF MIXED TERM. SO WE GET EITHER

$\pm(a^2x^2 + b^2y^2)$ $\qquad$ $\pm(a^2x^2 - b^2y^2)$

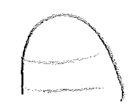
OR ONE OR MORE COEFF ARE 0

(OTHER CASES)

HOW DO WE TELL?

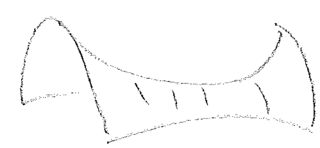
$$\text{DISCRIMINANT} \rightarrow D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} \leftarrow \text{DETERMINANT}$$

IF $D > 0$

$f_{xx} < 0$ LOCAL MAX 

$f_{xx} > 0$ LOCAL MIN 

$D < 0$ SADDLE POINT



$D = 0$ INCONCLUSIVE.

E.G. $f(x,y) = x^3 + x^2y^2 - y^4$

CRIT POINTS $(0,0)$, $\left(-3, \frac{3}{\sqrt{2}}\right)$, $\left(-3, \frac{-3}{\sqrt{2}}\right)$

$f_x = 3x^2 + 2xy^2 \qquad f_y = 2x^2y - 4y^3$

$f_{xx} = 6x + 2y^2 \qquad f_{yy} = 2x^2 - 12y^2$

$f_{xy} = f_{yx} = 2x^2 - 12y^2$

$(0,0) \qquad D = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$ INCONCLUSIVE

$\left(-3, \frac{3}{\sqrt{2}}\right) \qquad D = \begin{vmatrix} -9 & -\frac{36}{\sqrt{2}} \\ -\frac{36}{\sqrt{2}} & -36 \end{vmatrix} = 9 \cdot 36 - \frac{36^2}{2} < 0$

SADDLE POINT

$\left(-3, \frac{-3}{\sqrt{2}}\right)$ ALSO A SADDLE.

CONSIDERING THE BOUNDARY: ABSOLUTE MAX/MIN

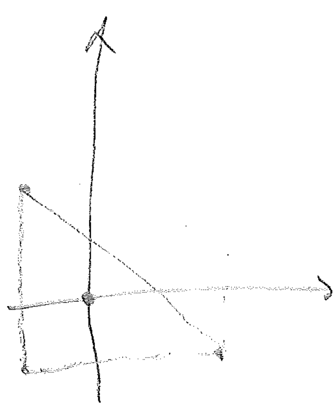
E.G.

FIND THE ABSOLUTE MAX/MIN OF $f(x,y) = x^2 + 3xy - 2y^2$ ON THE TRIANGLE T WITH VERTICES $(-1,2)$, $(-1,-1)$ AND $(2,-1)$.

i) FIND CRIT PTS:

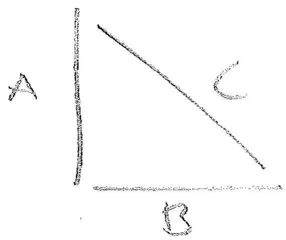
$f_x = 2x + 3y \qquad f_y = 3x - 4y$

$$\begin{cases} 2x + 3y = 0 \\ 3x - 4y = 0 \end{cases} \sim \begin{cases} x = 0 \\ y = \frac{3}{4}x \end{cases} \qquad (0,0)$$



$(0,0)$ IS IN THE DOMAIN.

ii) CHECK THE EDGES



A: $x = -1 \quad -1 \le y \le 2$

$f(-1, y) = -2y^2 - 3y + 1$

$f'(-1, y) = -4y - 3$ CRIT PT $y = -\frac{3}{4}$

IS WITHIN THE DOMAIN, SO WE ADD IT TO OUR LIST

B: $y = -1, \quad -1 \le x \le 2$

$f(x, -1) = x^2 - 3x - 2 \qquad f'(x, -1) = 2x - 3$

$x = -\frac{2}{3}$ WITHIN DOMAIN

C: $y = -x + 1 \quad -1 \le x \le 2$

$f(x, -x+1) = \ldots = -4x^2 + 7x - 2$

$f'(x, -x+1) = -8x + 7 \qquad x = \frac{7}{8}$ WITHIN DOMAIN

SO OUR LIST IS

$(0,0)$ ← CRIT POINT ON INSIDE

$(-1,-3/4)$, $(3/2,-1)$, $(7/8,\frac{1}{8})$ ← CRIT POINTS EDGES

$(-1,2)$, $(-1,-1)$, $(2,-1)$ ← VERTICES

VALUES

$0, \frac{17}{8}, -\frac{17}{4}, \frac{17}{16}, -13, 2, -4$

SO MIN $= -13$ AT $(-1,2)$

MAX $= \frac{17}{8}$ AT $(-1,\frac{-3}{4})$

<u>E.G.</u> IF THE DIAGONAL OF A BOX IS 1 METER, WHAT'S THE MAIMUM POSSIBLE VOLUME?

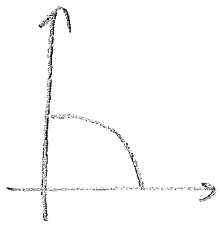
x, y, z SIDES

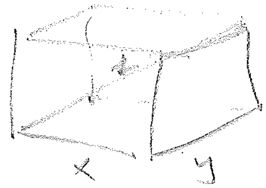
DIAG $= 1 = \sqrt{x^2+y^2+z^2}$

THEN $z = \sqrt{1-x^2-y^2}$

VOL $= xyz = xy\sqrt{1-x^2-y^2}$

DOMAIN $x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1$





$$V_x = y\sqrt{1-x^2-y^2} + \frac{\cancel{2}x^2y}{\cancel{2}\sqrt{1-x^2-y^2}}$$

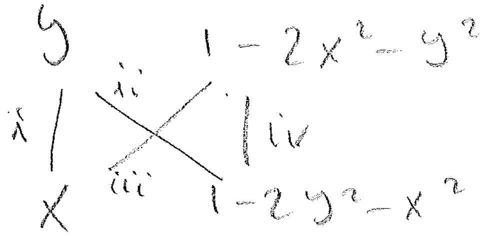
$$= \frac{y(1-x^2-y^2) + x^2y}{\sqrt{1-x^2-y^2}} = \frac{y-2yx^2-y^3}{\sqrt{1-x^2-y^2}}$$

$$V_y = \frac{x-2xy^2-x^3}{\sqrt{1-x^2-y^2}}$$

<u>CRIT PTS</u>: EITHER $\sqrt{1-x^2-y^2}=0$

$\Rightarrow$ BOUNDARY OR

$$\begin{cases} y-2yx^2-y^3=0 \\ x-2xy^2-x^3=0 \end{cases} \sim \begin{cases} y(1-2x^2-y^2)=0 \\ x(1-2y^2-x^2)=0 \end{cases}$$



i) $(0,0)$

ii) $y=0$

$1-x^2=0$ ON BOUNDARY

$x=1 \leftarrow (x \geq 0)$

iii) $x=0$

$1-y^2=0$ ON BOUNDARY

$y=1$

iv) $\begin{cases} 1-2x^2-y^2=0 \\ 1-2y^2-x^2=0 \end{cases} \wedge \begin{cases} y^2=1-2x^2 \\ -1+3x^2=0 \end{cases}$

$\begin{cases} y=\frac{1}{\sqrt{3}} \\ x=\frac{1}{\sqrt{3}} \end{cases} \geq 0$

NOTE: ON BOUNDARY

$V = xy\underbrace{\sqrt{1-x^2-y^2}}_{0} = 0$

SO MAX MUST BE AT $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

IN 3 VAR:

SAME IDEA. SAY YOU WANT

MAX / MIN $f(x)$ ON A BOX