SO OUR LIST IS

$(0,0)$ ← CRIT POINT ON INSIDE

$(-1,-3/4)$, $(3/2,-1)$, $(7/8, \frac{1}{8})$ ← CRIT POINTS EDGES

$(-1,2)$, $(-1,-1)$, $(2,-1)$ ← VERTICES

VALUES

$0, \frac{17}{8}, -\frac{17}{4}, \frac{17}{16}, -13, 2, -4$

SO MIN $= -13$ AT $(-1,2)$

MAX $= \frac{17}{8}$ AT $(-1, \frac{-3}{4})$

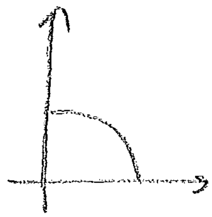
<u>E.G.</u> IF THE DIAGONAL OF A BOX IS 1 METER, WHAT'S THE MAXIMUM POSSIBLE VOLUME? x,y,z SIDES

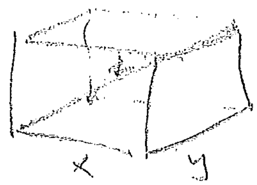
DIAG $= 1 = \sqrt{x^2+y^2+z^2}$

THEN $z = \sqrt{1-x^2-y^2}$

VOL $= xyz = xy\sqrt{1-x^2-y^2}$

DOMAIN $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq 1$





$$V_x = y\sqrt{1-x^2-y^2} + \frac{2x^2y}{2\sqrt{1-x^2-y^2}}$$

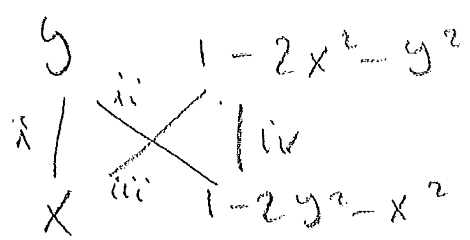
$$= \frac{y(1-x^2-y^2) + x^2y}{\sqrt{1-x^2-y^2}} = \frac{y - 2yx^2 - y^3}{\sqrt{1-x^2-y^2}}$$

$$V_y = \frac{x - 2xy^2 - x^3}{\sqrt{1-x^2-y^2}}$$

<u>CRIT PTS</u>: EITHER $\sqrt{1-x^2-y^2} = 0$

$\Rightarrow$ BOUNDARY OR

$$\begin{cases} y - 2yx^2 - y^3 = 0 \\ x - 2xy^2 - x^3 = 0 \end{cases} \sim \begin{cases} y(1-2x^2-y^2) = 0 \\ x(1-2y^2-x^2) = 0 \end{cases}$$



i) $(0,0)$

ii) $y = 0$

$1 - x^2 = 0$ ON BOUNDARY

$x = 1 \leftarrow (x \geq 0)$

iii) $x = 0$

$1 - y^2 = 0$ ON BOUNDARY

$y = 1$

iv) $\begin{cases} 1 - 2x^2 - y^2 = 0 \\ 1 - 2y^2 - x^2 = 0 \end{cases} \wedge \begin{cases} y^2 = 1 - 2x^2 \\ -1 + 3x^2 = 0 \end{cases}$

$\begin{cases} y = \frac{1}{\sqrt{3}} \\ x = \frac{1}{\sqrt{3}} \end{cases} \geq 0$

<u>NOTE</u>: ON BOUNDARY

$$V = xy \underbrace{\sqrt{1 - 2x^2 - y^2}}_{0} = 0$$

SO MAX MUST BE AT $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

IN 3 VAR:

SAME IDEA. SAY YOU WANT

MAX / MIN $f(x)$ ON A BOX

BOX $1 \le x \le 2$, $3 \le y \le 4$, $5 \le z \le 6$

1) FIND CRIT PTS

$f_x = f_y = f_z = 0$ INSIDE BOX

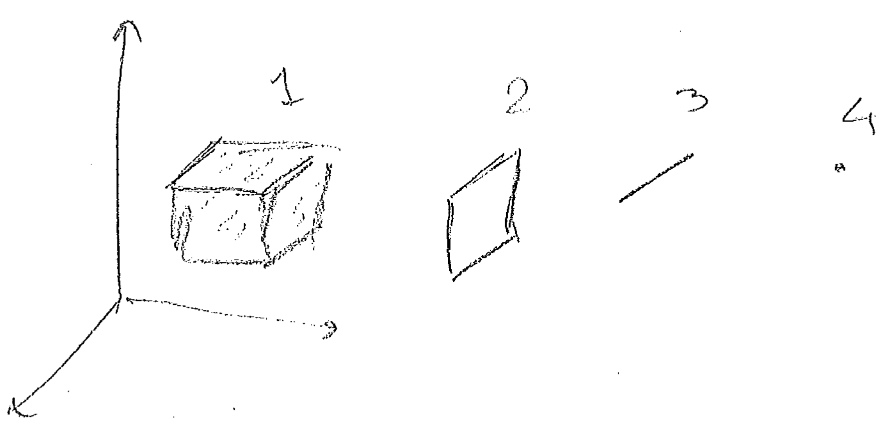
2) FACES OF BOX: PLUG IN

$x = 1$, GET f OF y, z, LOOK FOR CRIT PTS ... 6 FACES TOTAL

3) EDGES PLUG IN $x = 1$, $z = 5$, GET f OF y, LOOK FOR CRIT PTS...

12 CALCULATIONS

4) CHECK VERTICES



E.G.

FIND MIN/MAX OF $f(x,y,z)$

$= xy(1-z^2)$ ON $R = \{ z \geq 0 \quad x^2+y^2+z^2 \leq 1 \}$

$f_x = y(1-z^2) \quad f_y = x(1-z^2) \quad f_z = -2xyz$

$(*, *, \pm 1) \qquad (0, 0, *)$

BOUNDARY

- $z = 0 \qquad f(x,y) = xy \quad f_x = y, \; f_y = x$

  $(0, 0, 0)$

- $z = 1 - x^2 - y^2 \qquad f(x,y) = xy(x^2+y^2)$

  $f_x = y(3x^2+y^2) \qquad f_y = x(3y^2+x^2)$

  $(0, 0, 1) \qquad \begin{cases} 3x^2+y^2=0 \\ x^2+3y^2=0 \end{cases} \quad \begin{cases} y^2=0 \\ x^2=0 \end{cases}$

- $z = 0, \; x^2+y^2=1 \qquad f(x) = x\sqrt{1-x^2}$

  $f'(x) = \dfrac{1-2x^2}{\sqrt{1-x^2}} \qquad x = \pm\dfrac{1}{\sqrt{2}}$

$(*, *, \pm 1)$, $(0, 0, *)$ $f = 0$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ $f = \frac{1}{2}$

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ $f = \frac{1}{2}$

$\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$ $f = -\frac{1}{2}$

$\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ $f = -\frac{1}{2}$

MAX $\frac{1}{2}$, MIN $-\frac{1}{2}$

LAGRANGE MULTIPLIERS

WHAT IF THE BOUNDARY OF OUR REGION R IS COMPLICATED, OR DESCRIBED BY AN IMPLICIT FUNCTION?

E.G. FIND MAX/MIN OF

$f(x,y) = 2x + xy$ ON $\{x^2 + y^2 \leq 4\}$

INSIDE: $f_x = y + 2$, $f_y = x$

CRIT: $(0, -2)$

ON BOUNDARY:

TRICK BOUNDARY $g(x,y)=0$

$$\underset{x^2+y^2-4}{\uparrow}$$

WE LOOK FOR $\vec{\nabla} f = \lambda \vec{\nabla} g$

$$\underbrace{F(x,y,\lambda) = f(x,y) - \lambda g}$$

LOOK FOR CRIT PTS OF THIS

$$\begin{array}{l} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \lambda} \end{array} \begin{cases} y + 2 - 2\lambda x = 0 \\ x - 2\lambda y = 0 \\ x^2 + y^2 - 4 = 0 \end{cases} \sim \begin{cases} x^2 + 4\lambda^2 x^2 + 4 - 8\lambda x = 4 \\ y^2 + 4\lambda^2 y^2 = 4 \end{cases}$$

$$\begin{cases} x^2(4\lambda^2+1) = 8\lambda x \\ y^2(4\lambda^2+1) = 4 \end{cases} \qquad \begin{cases} x^2 = \dfrac{8\lambda x}{4\lambda^2+1} \\ y^2 = \dfrac{4}{4\lambda^2+1} \end{cases}$$

$$\frac{8\lambda x + 4}{4\lambda^2+1} = 4 \qquad 8\lambda x + \cancel{4} = 16\lambda^2 + \cancel{4}$$

$$\lambda x = 2\lambda^2 \qquad x = 2\lambda \quad y = 4\lambda^2 - 2$$

$$4\lambda^2 + 16\lambda^4 - 16\lambda^2 + 4 - 4 = 0 \qquad 16\lambda^4 - 12\lambda^2 = 0$$

$\lambda^2 = \frac{3}{4} \quad \lambda = \frac{\pm\sqrt{3}}{2}$

SO WE GET

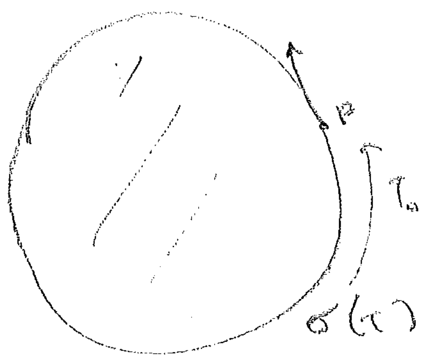
$x = \pm\sqrt{3}, \quad y = 1$

AT $(+\sqrt{3}, 1)$ $f(x,y) = 3\sqrt{3}$

AT $(-\sqrt{3}, 1)$ $f(x,y) = -3\sqrt{3}$

<u>WHY IT WORKS</u>

$g(x,y) = 0 \longleftrightarrow \sigma(t)$ LOCALLY



IF P IS A MAX OF

$f(\sigma(t))$ THEN $\frac{d}{dt} f(\sigma(t))\Big|_{t_0} = 0$

$$\vec{\nabla} f\big|_P \cdot \langle \underbrace{x'(t_0), y'(t_0)}_{\text{TANGENT VECTOR}} \rangle$$

SO $\vec{\nabla} f \perp$ TO TANGENT TO

$\sigma \Rightarrow$ IT'S NORMAL TO THE LEVEL

CURVE $\Rightarrow$ IT'S // TO $\vec{\nabla} g$

E.G. $f(x,y,z) = xy(1-z^2)$ ON UPPER HALF SPHERE OF $r=1$ WITH LAGRANGE.

NOTE THAT $f(x,y,z) = f(x,y,-z)$
SO MAX/MIN ON HALF SPHERE ARE SAME AS ON SPHERE $x^2+y^2+z^2=1$

$$F(x,y,z,\lambda) = xy(1-z^2) - \lambda(x^2+y^2+z^2-1)$$

$$\vec{\nabla} F = 0$$

$$\begin{cases} y(1-z^2) - 2\lambda x = 0 \\ x(1-z^2) - 2\lambda y = 0 \\ -2xyz - 2\lambda z = 0 \\ x^2+y^2+z^2 = 1 \end{cases} \qquad \begin{cases} \\ \\ -2z(\lambda + xy) = 0 \\ \\ \end{cases}$$

a.) $\lambda = -xy$

$x, y \neq 0$

$$\begin{cases} y(1-z^2) + 2x^2y = 0 \\ x(1-z^2) + 2y^2x = 0 \\ x^2+y^2+z^2 = 1 \end{cases} \qquad \begin{cases} y(2x^2-z^2+1) = 0 \\ x(2y^2-z^2+1) = 0 \\ x^2+y^2+2y^2+1 = 1 \end{cases}$$

$$\begin{cases} 2x^2 = 2y^2 \\ x^2 = -3y^2 \end{cases}$$ NO SOL.

$(0,0,*)$
ONLY SOL

b) $z = 0$

$$\begin{cases} y - 2\lambda x = 0 \\ x - 2\lambda y = 0 \\ x^2 + y^2 = 1 \end{cases} \qquad \begin{cases} y - 4\lambda^2 y = 0 \\ \\ \\ \end{cases} \qquad \begin{cases} \lambda = \pm \frac{1}{2} \\ x \pm y = 0 \\ 2x^2 = 1 \end{cases}$$

$$\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, 0 \right)$$

<u>E.G.</u> $f(x, y, z) = xyz$ ON ELLIPSOID

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$$

$$F(x, y, z, \lambda) = f - \lambda\left(x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 1\right)$$

$$\vec{\nabla} F = 0$$

$$\begin{cases} yz - 2\lambda x = 0 \\ xz - \frac{\lambda y}{2} = 0 \\ xy - \frac{2\lambda z}{9} = 0 \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1 \end{cases} \qquad \begin{cases} yz = 2\lambda x \\ \frac{x}{y} = \frac{y}{4x} \sim 4x^2 = y^2 \\ \frac{y}{z} = \frac{4}{9}\frac{z}{y} \sim \frac{9}{4}y^2 = z^2 \\ \frac{z^2}{9} + \frac{z^2}{9} + \frac{z^2}{9} = 1 \end{cases}$$

i) $\lambda \neq 0$ or $f \neq 0$ ii) $x, y, z \neq 0$ or $f = 0$

$z = \pm\sqrt{3}$ $\quad y = \pm\frac{2}{\sqrt{3}}$ $\quad x = \pm\frac{1}{\sqrt{3}}$ $\qquad$ MAX $\frac{2}{\sqrt{3}}$

MIN $-\frac{2}{\sqrt{3}}$