E.G. $f(x,y) = x^3y^2 + 5y^2 - x + 7$

$$f_x(x,y) = \frac{\partial}{\partial x}(x^3y^2) + \frac{\partial}{\partial x}(5y^2) + \frac{\partial}{\partial x}(-x+7)$$

$$= 3x^2y^2 + 0 + (-1) = 3x^2y^2 - 1$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(y^2(x^3+5)\right) + \frac{\partial}{\partial y}(-x+7) =$$

$$2y(x^3+5).$$

HIGHER ORDER PARTIAL DERIVATIVES

GIVEN A FUNCTION $f(x,y)$ (OR $f(x,y,z)$) THE PARTIAL DERIVATIVES f_x, f_y (AND f_z) ARE STILL FUNCTIONS IN THE SAME NUMBER OF VARIABLES, SO WE CAN DIFFERENTIATE THEM AGAIN, WITH RESPECT TO ANY VAR.

WE GET:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad \underbrace{f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \qquad f_{xz} = \frac{\partial^2 f}{\partial z \partial x}}_{\text{THESE ARE CALLED "MIXED"}} \qquad \ldots$$

SECOND ORDER PARTIAL DERIVATIVES.

E.G.

$$f(x,y,z) = \sin(x^2y - z) \quad \text{AS BEFORE}$$

$$f_{xx} = \frac{\partial}{\partial x} 2xy\cos(x^2y-z) = \leftarrow \text{PRODUCT + CHAIN RULE}$$

$$2y\cos(x^2y-z) - 4x^2y^2\sin(x^2y-z)$$

$$f_{xz} = \frac{\partial}{\partial z} 2xy\cos(x^2y-z) = 2xy\frac{\partial}{\partial x}\left(\cos(x^2y-z)\right)$$

$$= 2xy(-1)(-\sin(x^2y-z)) = 2xy\sin(x^2y-z)$$

$$f_{zx} = \frac{\partial}{\partial x}\left(-\cos(x^2y-z)\right) = 2xy\sin(x^2y-z)$$

SAME ?!

E.G. $f(x,y) = x^3y^2 + 5y^2 - x + 7$

$$f_{xy} = (3x^2y^2 - 1)_y = 6x^2y$$

$$f_{yx} = (2y(x^3+5))_x = 6x^2y$$

SAME!

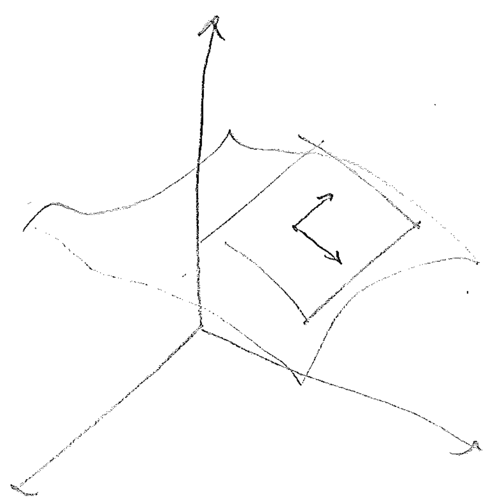
IS IT ALWAYS TRUE THAT ORDER DOES NOT COUNT IN MIXED PARTIAL DERIVATIVES? <u>ALMOST</u>

<u>THM</u>: IF $f(x,y,z)$ (OR $f(x,y)$) IS "NICE ENOUGH" THEN MIXED PARTIAL DERIVATIVES ARE INDEPENDENT OF ORDER OF DIFFERENTIATION.

"NICE ENOUGH": THE MIXED DERIV. ALL EXIST AND ARE CONTINUOUS (CLOSE TO A POINT).

GEOMETRIC INTERPRETATION

IN 2 VARIABLES,



$f_x(a,b)$, $f_y(a,b)$ ARE THE SLOPES OF THE TANGENT PLANE TO THE GRAPH OF $f(x,y)$ AT (a,b) IN THE x AND y DIRECTIONS.

E.G.

IS THERE AN $f(x,y)$ WITH:

$$\begin{cases} f_x = x^3y + y^2 - x \\ f_y = x^4 + 2xy \end{cases}$$

THEN $f_{xy} = x^3 + 2y$

$f_{yx} = 2x$

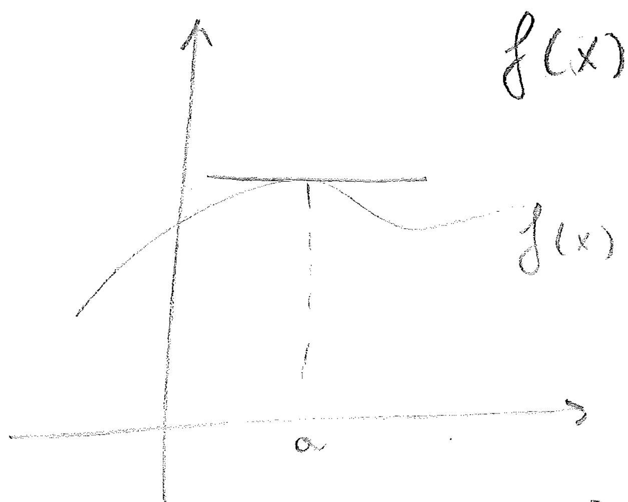
BUT BOTH EXIST AND ARE CONTINUOUS, SO THEY SHOULD BE EQUAL!

LINEAR APPROXIMATIONS AND TANGENT PLANES

RECALL: 1 VAR

CLOSE TO a

$$f(x) \approx f(a) + f'(a)(x-a)$$



TANGENT LINE AT $(a, f(a))$

$$y = f(a) + f'(a)(x-a)$$

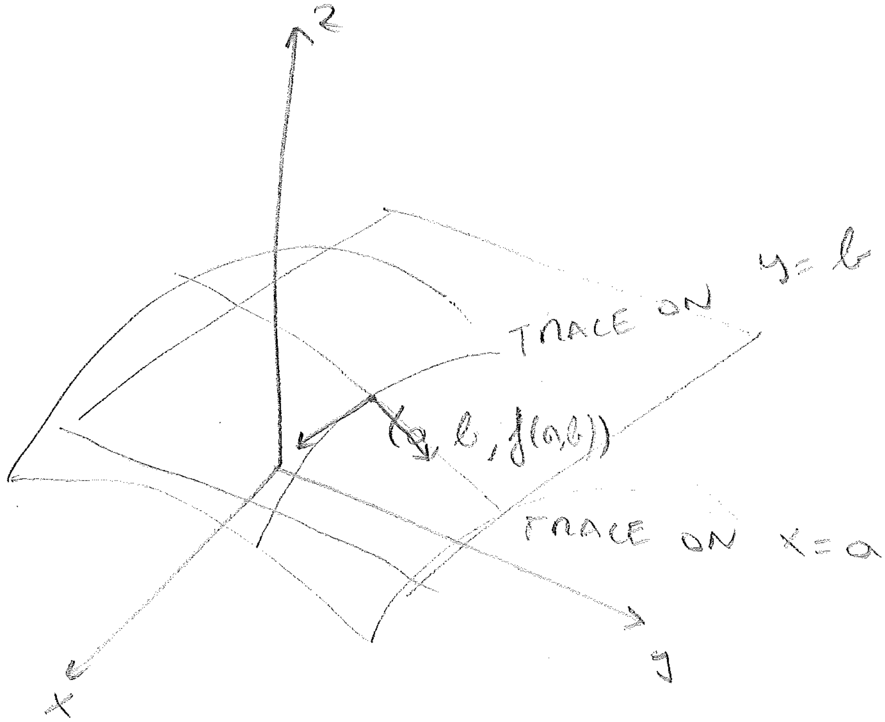
$f'(a) =$ SLOPE

IN 2 VAR:

$f(x,y)$, WE WANT A LINEAR APPROX NEAR POINT (a,b)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = L(x,y)$$ LINEAR FUNCTION

THE GRAPH $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ IS THE TANGENT PLANE TO THE GRAPH OF $f(x,y)$ AT $(a,b,f(a,b))$



E.G.

FIND AN EQUATION TO THE TANGENT PLANE TO THE GRAPH OF THE FUNCTION

$f(x,y) = x^2 - 3xy$ AT THE POINT $(1,2,-5)$.

$f_x = 2x - 3y \qquad f_y = -3x$

AROUND $(1,2)$

$$f(x,y) \approx -5 + (-4)(x-1) + (-3)(y-2)$$

$$z = (-5+4+6) - 4x - 3y = 5 - 4x - 3y$$

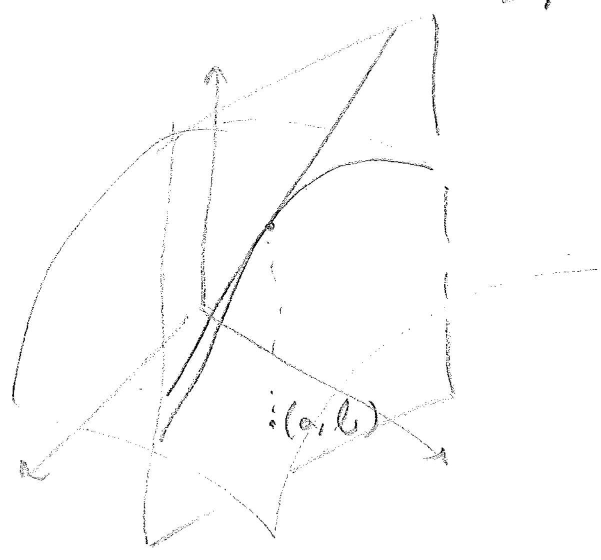
(SANITY CHECK: VERIFY $(1,2,-5)$
$-5 = 5 - 4\cdot 1 - 3\cdot 2 = 5 - 10$ ✓)

QUICK DISCUSSION: WHY IS THIS REALLY THE TANGENT PLANE?

LINE TANGENT TO TRACE AT $y = b$

SLOPE IS

$$\frac{\partial}{\partial x} f \Big|_{(a,b)}$$



SO THE LINE IS

$$\langle a, b, f(a,b)\rangle + t\langle 1, 0, f_x(a,b)\rangle$$

LINE TANGENT TO TRACE AT $x = a$

$$\langle a, b, f(a,b)\rangle + t\langle 0, 1, f_y(a,b)\rangle$$

NORMAL TO PLANE

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x(a,b), -f_y(a,b), 1\rangle = \vec{N}$$

EQUATION

$$\langle x, y, z\rangle \cdot \vec{N} = \langle a, b, f(a,b)\rangle \cdot \vec{N}$$

$$-f_x(a,b)\,x - f_y(a,b)\,y + z = -a f_x(a,b) - b f_y(a,b) + f(a,b)$$

$$z = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

SAME EQUATION!