A BRIEF DISCUSSION OF CONTINUITY IN 2 VARIABLES

RECALL:

A FUNCTION $f(x)$ IN ONE VARIABLE IS **CONTINUOUS** AT a IF $\lim_{x \to a} f(x) = f(a)$

OFTEN WE HAVE A FUNCTION $f(x)$ DEFINED BY A FORMULA WHICH MAKES NO SENSE AT A POINT a, AND WE WANT TO SEE IF IT CAN BE CONTINUOUSLY EXTENDED AT a.

E.G.

- $f(x) = \frac{x^2-1}{x-1}$ $a=1$ $\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$

$= 2$ ✓

- $f(x) = \frac{\sin x}{x}$ $a=0$ $\lim_{x \to 0} \frac{\sin x}{x} \overset{\text{TAYLOR}}{=}$

$\lim_{x \to 0} \frac{x - \frac{x^3}{6} + \dots}{x} = 1$ ✓

- $f(x) = \frac{\log(x)}{(x-1)^2}$ $a=1$ $\lim_{x \to 1} \frac{\log x}{(x-1)^2} \overset{\text{TAYLOR}}{=}$

$\lim_{x \to 1} \frac{(x-1) - \frac{(x-1)^2}{2} + \dots}{(x-1)^2}$ DNE ✗

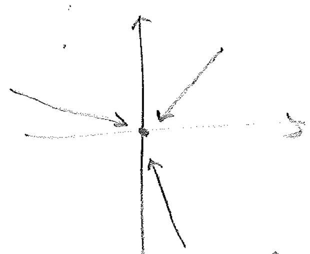
IN TWO VARIABLES:

WHILE IN ONE DIMENTION YOU CAN BASICALLY "APPROACH" A POINT IN TWO WAYS (FROM EITHER SIDE) IN TWO DIMENSIONS THERE ARE INFINITELY MANY WAYS TO APPROACH, AND EACH MIGHT GIVE A DIFFERENT RESULT!

E.G. $$f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

FORMULA IS NOT DEFINED AT $O = (0,0)$

- APPROCHING O ALONG LINES $y = mx$



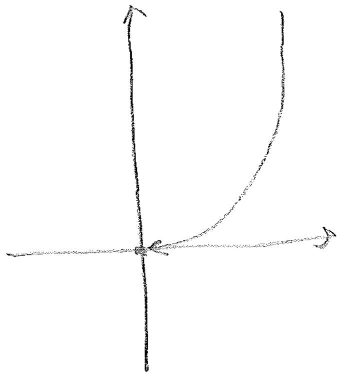
$$f(x, mx) = \frac{x^2 (mx)}{x^4 + (mx)^2}$$

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{mx^3}{x^4 + m^2 x^2}$$

$$\lim_{x \to 0} \underbrace{\frac{x^2}{x^2}}_{=1} \cdot \frac{\overbrace{mx}^{\to 0}}{\underbrace{m^2 + x^2}_{\to m^2}} = 0$$

- APPROACHING O ALONG PARABOLAS $y = mx^2$

$$f(x, mx^2) = \frac{mx^4}{x^4 + m^2 x^4}$$



$$\lim_{x \to 0} f(x, mx^2) = \lim_{x \to 0} \frac{mx^4}{(m^2+1)x^4}$$

$$= \frac{m}{m^2+1} \neq 0 \,!$$

SO $f(x,y)$ CANNO BE EXTENDED TO A CONTINUOUS FUNCTION AT $(0,0)$.

<u>TAKEOUT</u>: CONTINUITY IN MORE THAN 1 VARIABLE IS HARD! LUCKILY, AS FOR 1 VARIABLE, WHEN $f(x,y)$ IS DEFINED BY SOME FORMULA IN THE "NATURAL" FUNCTIONS (POLYNOMIALS, TRIG FUNCTIONS, EXPONENTIALS, LOGARITHMS...) IT WILL BE CONTINUOUS WHEREVER THE FORMULA MAKES SENSE.

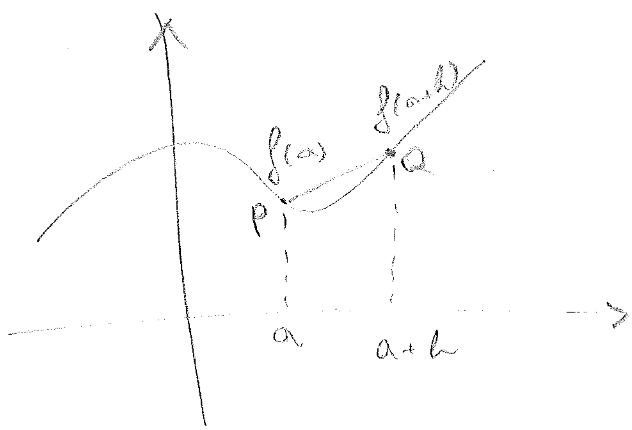
<u>E.G.</u> (OPTIONAL): $f(x,y) = \dfrac{5x^2y^2}{x^2+y^2}$ AT $(0,0)$

WE WANT $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$.

NOTE THAT $y^2 \leq x^2+y^2$, SO $f(x,y) \leq 5x^2 \cdot \left(\dfrac{y^2}{x^2+y^2}\right) \leq 5x^2$. SO $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

PARTIAL DERIVATIVES

IN 1 VARIABLE



$$\left.\frac{df}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \underbrace{\frac{f(a+h)-f(a)}{h}}_{\text{SLOPE OF SEGMENT } \overline{PQ}}$$

= SLOPE OF TANGENT LINE AT a = RATE OF CHANGE OF f AT a.

WE WANT AN ANALOGUE CONCEPT IN MORE THAN ONE VARIABLE.

IDEA: FIX ALL VARIABLES EXCEPT ONE, FIND RATE OF CHANGE OF THE CORRESPONDING 1-VARIABLE FUNCTION

DEFINITION:

2 VARIABLES: $f(x,y)$ FUNCTION IN 2 VAR, THEN

DIFF. SYMBOL →

$$\underbrace{\left.\frac{\partial}{\partial x} f\right|_{(a,b)} = f_x(a,b)}_{\text{2 NOTATIONS}} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$= \frac{d}{dx} f(x,b) \Big|_{x=a} \leftarrow \text{FIX } y=b, \text{ NOW } f(x,b) \text{ IS 1-VAR}$$

$$\frac{d}{dy} f \Big|_{(a,b)} = f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

$$= \frac{d}{dy} f(a,y) \Big|_{y=b}$$

3 VARIABLES: SAME IDEA

$$\frac{\partial}{\partial x} f(x,y,z) \Big|_{(a,b,c)} = \lim_{h \to 0} \frac{f(a+h,b,c)-f(a,b,c)}{h}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial}{\partial z} f(x,y,z) \Big|_{(a,b,c)} = \lim_{h \to 0} \frac{f(a,b,c+h)-f(a,b,c)}{h}$$

HOW TO COMPUTE, IN PRACTICE:

PRETEND ALL VARIABLES EXCEPT THE ONE YOU ARE DERIVING W.R.T. ARE CONSTANT, TAKE A NORMAL DERIVATIVE.

E.G. $f(x,y) = y e^{x+2y}$

$$f_x(x,y) = \frac{\partial}{\partial x} y e^{x+2y} = \overbrace{y e^{2y}}^{\text{TREAT AS CONSTANT}} \frac{\partial}{\partial x} e^{x} =$$

$$y e^{2y+x}$$

$$f_y(x,y) = \frac{\partial}{\partial y} y e^{x+2y} = e^{x} \frac{\partial}{\partial y} y e^{2y} =$$

$$e^{x+2y} + 2y e^{x+2y}$$

- FIND $f_x(3,5)$, $f_y(3,5)$

$$f_x(3,5) = 5 e^{3+2\cdot 5} = 5 e^{13}$$

$$f_y(3,5) = e^{3+10} + 2\cdot 5\, e^{3+10} = 11 e^{13}$$

E.G. $f(x,y,z) = \sin(x^2 y - z)$

$$f_x(x,y,z) = \frac{\partial}{\partial x} \sin(x^2 y - z) \underset{\uparrow}{=}$$

CHAIN RULE

$$\left(\frac{\partial}{\partial x}(x^2 y - z)\right)\cos(x^2 y - z) = 2xy\cos(x^2 y - z)$$

$$f_y(x,y,z) = \frac{\partial}{\partial y} \sin(x^2 y - z) = \frac{\partial}{\partial y}(x^2 y - z)\cos(x^2 y - z)$$

$$= x^2 \cos(x^2 y - z)$$

$$f_z(x,y,z) = \frac{\partial}{\partial z} \sin(x^2 y - z) = \frac{\partial}{\partial z}(x^2 y - z)\cos(x^2 y - z)$$

$$= -\cos(x^2 y - z)$$

E.G. $f(x,y) = x^3y^2 + 5y^2 - x + 7$

$$f_x(x,y) = \frac{\partial}{\partial x}(x^3y^2) + \frac{\partial}{\partial x}(5y^2) + \frac{\partial}{\partial x}(-x+7)$$

$$= 3x^2y^2 + 0 + (-1) = 3x^2y^2 - 1$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left(y^2(x^3+5)\right) + \frac{\partial}{\partial y}(-x+7) =$$

$$2y(x^3+5).$$

HIGHER ORDER PARTIAL DERIVATIVES

GIVEN A FUNCTION $f(x,y)$ (OR $f(x,y,z)$) THE PARTIAL DERIVATIVES f_x, f_y (AND f_z) ARE STILL FUNCTIONS IN THE SAME NUMBER OF VARIABLES, SO WE CAN DIFFERENTIATE THEM AGAIN, WITH RESPECT TO ANY VAR.

WE GET:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad \underbrace{f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{xz} = \frac{\partial^2 f}{\partial x \partial z}}_{\text{THESE ARE CALLED "MIXED"}} \qquad \ldots$$

SECOND ORDER PARTIAL DERIVATIVES.