WORKSHEET (MORE THEORETICAL) ?

1) THE WAVE EQUATION IN ONE SPACE DIMENSION IS: $u = u(x,t)$

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x} \qquad (u_{tt} = c^2 u_{xx})$$

WHERE c IS A CONSTANT (FOR EXAMPLE THE SPEED OF LIGHT FOR ELECTROMAGNETIC WAVES).

PROVE THAT ANY FUNCTION IN THIS FORM

$$u(x,t) = f(x-ct) + g(x+ct)$$

WHERE f, g ARE 1 VAR. SATISFIES THE EQUATION.

2) CAN WE FIND A $f(x,y)$ SUCH THAT $f_x(x,y) = x^3y + y^2 x$, $f_y(x,y) = x^4 + 2xy$? WHAT ABOUT $f_x(x,y) = 4x^3y + y^2 - x$, $f_y(x,y) = x^4 + 2xy$?

SOL:

1) $$u(x,T) = f(x-eT) + g(x+eT)$$

$$u_x(x,T) = f'(x-eT) + g'(x+eT)$$

$$u_{xx}(x,T) = f''(x-eT) + g''(x+eT)$$

$$u_T(x,T) = -e f'(x-eT) + e g'(x+eT)$$

$$u_{TT}(x,T) = e^2 f''(x-eT) + e^2 g'(x+eT)$$

$$= e^2 u_{xx}(x,T).$$

CONCRETE EXAMPLE:

$$u(x,T) = \cos(x-eT) + (x+eT)^2$$

$$u_{xx}(x,T) = -\cos(x-eT) + 2$$

$$u_{TT}(x,T) = -e^2\cos(x-eT) + 2e^2$$

2) FIRST QUESTION:

DOES SUCH f EXIST? $\begin{cases} f_x = x^3y + y^2 - x \\ f_y = x^4 + 2xy \end{cases}$

IDEA: INTEGRATE W.R.T. x, COMPARE

ANTI DERIVATIVE $\int F_x \, dx = \frac{x^4}{4}y + xy^2 - \frac{x^2}{2} + C(y)$

CAN DEPEND ON OTHER VARIABLE!

$$\frac{\partial F_x}{\partial y} = \frac{x^4}{4} + 2xy + C'(y)$$

$$\frac{\partial F_x}{\partial y} - f_y = \frac{-3x^4}{4} + C'(y)$$

WE CANNOT CANCEL THE x^4 TERM, SO THE ANSWER IS NO!

NOTE: WE COULD HAVE FOUND ANTI-DER OF f_y, COMPARED $\frac{\partial F_y}{\partial x}$ WITH f_x, OR FOUND BOTH F_x, F_y AND COMPARED THEM.

SECOND QUESTION:

$$\begin{cases} f_x = 4x^3y + y^2 - x \\ f_y = x^4 + 2xy \end{cases}$$

$$F_x = x^4y + xy^2 - \frac{x^2}{2} + C(y)$$

$$\frac{\partial F_x}{\partial y} = x^4 + 2xy + C'(y)$$

$$\frac{\partial F_x}{\partial y} - f_y = C'(y)$$ SO WE CAN JUST PICK $C'(y)$ CONSTANT AND ANSWER IS YES.

THE TOTAL DIFFERENTIAL

$f(x,y)$ FUNCTION OF 2 VARIABLES

<u>THE TOTAL DIFFERENTIAL</u> OF $f(x,y)$

AT (a,b) IS:

$$df = f_x(a,b)\,dx + f_y(a,b)\,dy$$

dx, dy ARE FORMAL SYMBOLS. THEY ARE "THE SAME" AS THE ONES UNDER THE INTEGRAL SYMBOL.

$f(x,y,z)$ FUNCTION OF 3 VAR:

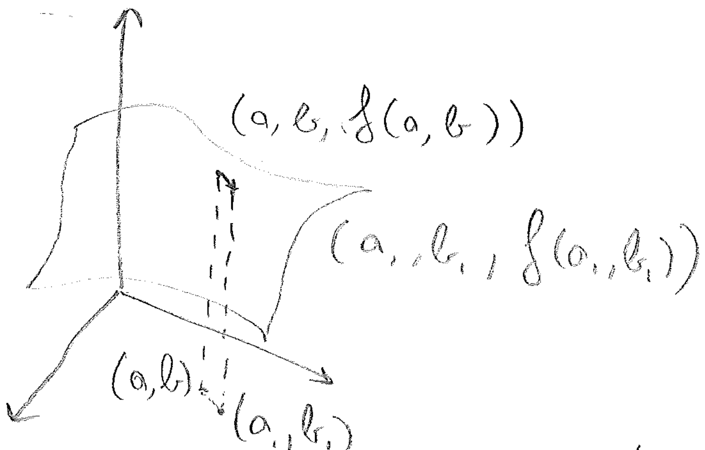
$$df = f_x(a,b,c)\,dx + f_y(a,b,c)\,dy + f_z(a,b,c)\,dz$$

INTERPRETATION:

$$\Delta f \approx f_x(a,b,c)\Delta_x + f_y(a,b,c)\Delta_y + f_z(a,b,c)\Delta_z$$

$$\Delta f = f(a_1,b_1,c_1) - f(a,b,c)$$

FOR a_1, b_1, c_1 CLOSE TO a, b, c $(\Delta_x = a_1 - a;\ \Delta_y = b_1 - b,\ \Delta_z = c_1 - c)$



DIFFERENCE

$$f(a_1, b_1) - f(a_1, b_1)$$

IS APPROX

$$(a_1 - a) f_x(a, b) + (b_1 - b) f_y(a, b)$$

WHEN $a_1 - a \,,\, b_1 - b \sim 0$

E.G. WE WANT TO FIND THE VOLUME OF A BOX. WE MEASURED

- LENGTH $l = 10$ UP TO 1 mm
- WIDTH $w = 15$ UP TO 2 mm
- HEIGTH $h = 12$ UP TO $.3$ mm

ESTIMATE THE POSSIBLE ERROR IN VOLUME.

$$V(l, w, h) = l \cdot w \cdot h$$

$$dV = wh\,dl + lh\,dw + lw\,dh$$

$$\Delta V = wh\,\Delta l + lh\,\Delta w + lw\,\Delta h$$

$$= 15 \cdot 12 \cdot \underbrace{1}_{\text{ERROR IN } l} + 10 \cdot 12 \cdot \underbrace{2}_{\text{ERROR IN } w} + 10 . 15 \cdot \underbrace{(.3)}_{\text{ERROR IN } h} = 465 \text{ mm}^3$$

IMPLICIT DIFFERENTIATION

AN EQUATION $F(x,y,z)=0$ IN THREE VARIABLES DEFINES "IMPLICIT" FUNCTIONS GIVEN BY SOLVING FOR z.

JUST LIKE WE DID FOR $F(x,y)=0$ (WHICH IMPLICITLY DEFINES A ONE-VAR FUNCTION) WE CAN FIND INFORMATION ON THE PARTIAL DERIVATIVES OF THIS FUNCTIONS BY MANIPULATING THE EQUATION.

E.G. $x^2+3y^2+5z^2-58=0$ (ELLIPSOID)

FIXING x,y THERE ARE 0, 1, OR 2 SOLUTIONS FOR z

EXPLICITLY: $z = \pm\sqrt{58-x^2-3y^2}\cdot\frac{1}{\sqrt{5}}$

BUT SOMETIMES WE CANNOT EXPLICITLY SOLVE $F(x,y,z)=0$! HOW DO WE GET INFORMATION?

$z=f(x,y)$ FIND $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

AT $(1,2,3)$ (NOTE: $(1,2,3)$ SOLVES EQ.)

IDEA: DIFFERENTIATE TREATING z AS FUNCTION

$$x^2 + 3y^2 + 5z^2 - 58 = 0$$

$$\frac{\partial}{\partial x}\left(x^2 + 3y^2 + 5z^2 - 58\right) = \frac{\partial}{\partial x}(0)$$

$$2x + 10z \cdot z_x = 0 \qquad z_x = -\frac{x}{5z}$$

AT $(1,2,3)$ $\left.\frac{\partial z}{\partial x}\right|_{(1,2,3)} = -\frac{1}{15}$

$$\frac{\partial}{\partial y}\left(x^2 + 3y^2 + 5z^2 - 58\right) = 0$$

$$6y + 10z\,z_y = 0 \qquad z_y = -\frac{3y}{5z}$$

AT $(1,2,3)$ $\left.\frac{\partial z}{\partial y}\right|_{(1,2,3)} = -\frac{2}{5}$

E.G. SURFACE $x^3y + z^3x + y^3z = 3$

POINT $(1,1,1)$.

APPROXIMATE THE VALUE OF z WHEN

$x = 1.01,\ y = 1.01$

WE WANT TO USE A LINEAR APPROXIMATION

$$\frac{\partial}{\partial x}\left(x^3y + z^3x + y^3z\right) = 0$$

$$3x^2y + 3z^2 z_x x + z^3 + y^3 z_x = 0$$

$$z_x = \frac{-3x^2y - z^3}{3z^2 + y^3}$$

AT $(1,1,1)$ $\quad z_x = \frac{-4}{4} = -1$

$$\frac{\partial}{\partial y}\left(x^3y + z^3x + y^3z\right) = 0$$

$$x^3 + 3y^2z + y^3 z_y + 3z^2 z_x x = 0$$

$$z_y = \frac{-x^3 - 3y^2z}{y^3 + 3z^2 z_x x}$$

AT $(1,1,1)$ $\quad z_y = \frac{-4}{4} = -1$

SO $\quad L(x,y) = 1 - (x-1) - (y-1)$

$$L(1.01, 1.01) = 1 - \frac{2}{100} = \frac{98}{100}$$

SO $\quad z(1.05, 1.05) \approx 0.98$

(ACTUAL VALUE $z = 0.9\cancel{7}9\ldots$)

CHAIN RULE

1 VAR: X AS FUNCTION OF

$$\frac{d}{d\tau} f(x(\tau)) = x'(\tau) f'(x(\tau)) \quad \left(\begin{matrix} dx = \\ x' \, \dfrac{dx}{d\tau} \end{matrix} \right)$$

NOW SUPPOSE THAT BOTH X AND Y DEPEND ON τ (SAY, WE ARE FOLLOWING A PARTICLE MOVING ALONG A CURVE $\sigma(\tau) = (x(\tau), y(\tau))$) AND WE WANT TO MEASURE THE CHANGE IN A FUNCTION $f(x, y)$ AS A FUNCTION OF τ. HOW DOES THIS WORK?

$$\begin{tikzcd}[column sep=small] & f \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ \tau & & \tau \end{tikzcd}$$

FUNCTION OF

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x} \cdot \frac{dx}{d\tau} + \frac{\partial f}{\partial y} \cdot \frac{dy}{d\tau}$$

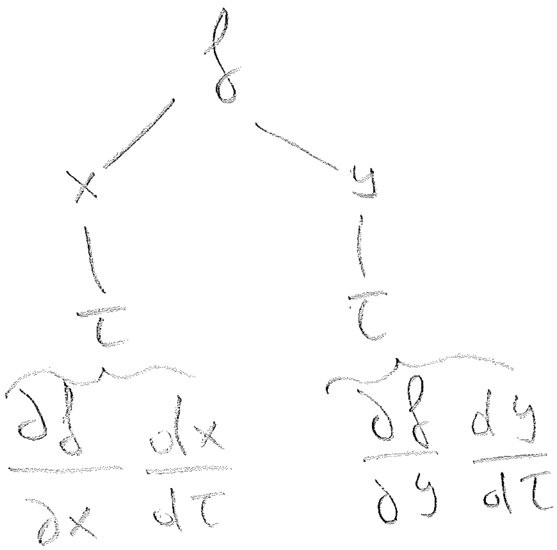
"WHY": $df = \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy$

TOTAL DER.

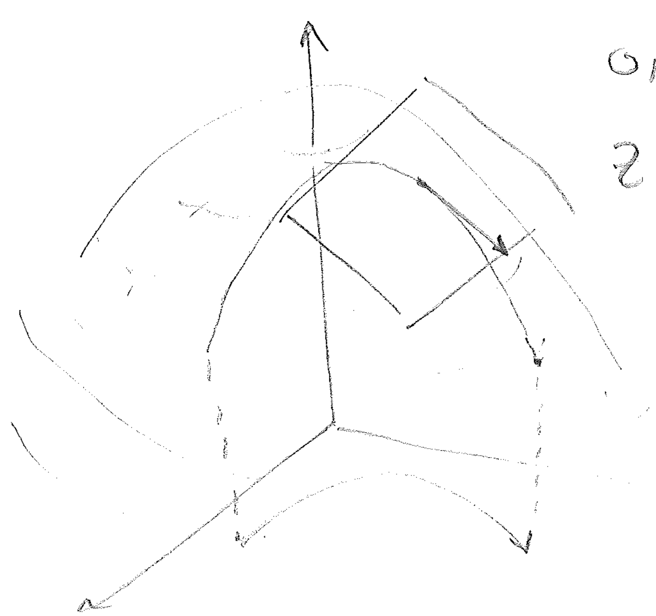
$$dx = \frac{dx}{d\tau}\, d\tau \qquad dy = \frac{dy}{d\tau}\, d\tau$$

$$df = \frac{\partial f}{\partial x}\frac{dx}{d\tau}\, d\tau + \frac{\partial f}{\partial y}\frac{dy}{d\tau}\, d\tau$$

TO REMEMBER: USE THE "FUNCTION OF" DIAGRAM



GEOMETRICALLY:



POINT MOVING ON SURFACE $z = f(x,y)$ WITH EQUATION

$$\big(x(\tau), y(\tau), f(x(\tau), y(\tau))\big)$$

VELOCITY VECTOR

$$\left\langle x'(\tau), y'(\tau), \frac{\partial f}{\partial x} x'(\tau) + \frac{\partial f}{\partial y} y'(\tau) \right\rangle$$