CHAIN RULE

1 VAR: X AS FUNCTION OF

$$\frac{d}{d\tau} f(x(\tau)) = x'(\tau) f'(x(\tau)) \quad \left(\begin{matrix} dx = \\ x' \frac{dx}{d\tau} \end{matrix} \right)$$

NOW SUPPOSE THAT BOTH x AND y DEPEND ON τ (SAY, WE ARE FOLLOWING A PARTICLE MOVING ALONG A CURVE $\sigma(\tau) = (x(\tau), y(\tau))$) AND WE WANT TO MEASURE THE CHANGE IN A FUNCTION $f(x, y)$ AS A FUNCTION OF τ. HOW DOES THIS WORK?

FUNCTION OF

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ \tau & & \tau \end{tikzcd}$$

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x} \cdot \frac{dx}{d\tau} + \frac{\partial f}{\partial y} \cdot \frac{dy}{d\tau}$$

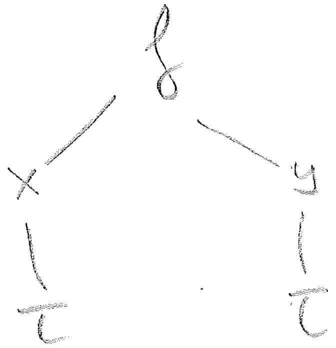
"WHY": $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

TOTAL DER.

$$dx = \frac{dx}{d\tau}\, d\tau \qquad dy = \frac{dy}{d\tau}\, d\tau$$

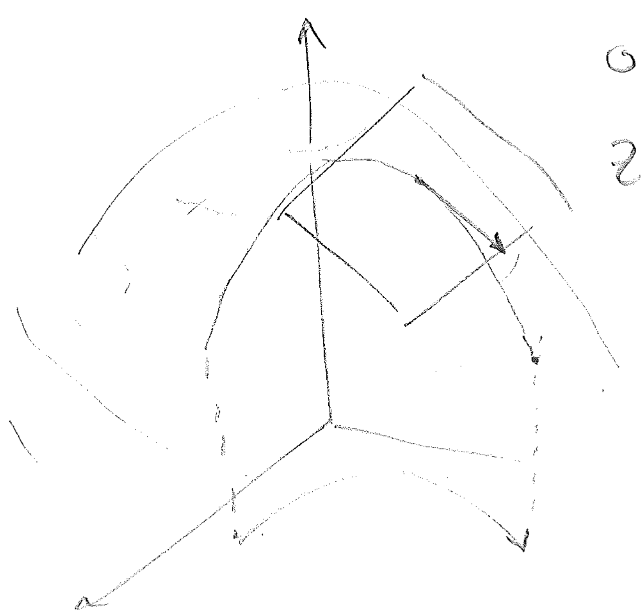
$$df = \frac{\partial f}{\partial x}\frac{dx}{d\tau}\, d\tau + \frac{\partial f}{\partial y}\frac{dy}{d\tau}\, d\tau$$

TO REMEMBER: USE THE "FUNCTION OF" DIAGRAM



$$\frac{\partial f}{\partial x}\frac{dx}{d\tau} \qquad \frac{\partial f}{\partial y}\frac{dy}{d\tau}$$

GEOMETRICALLY:



POINT MOVING ON SURFACE $z = f(x,y)$ WITH EQUATION

$$(x(\tau), y(\tau), f(x(\tau), y(\tau)))$$

VELOCITY VECTOR

$$\left\langle x'(\tau), y'(\tau), \frac{\partial f}{\partial x}x'(\tau) + \frac{\partial f}{\partial y}y'(\tau) \right\rangle$$

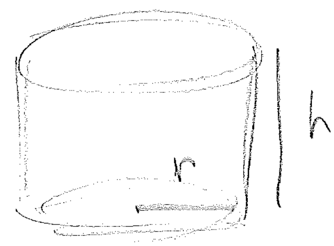
$$= \left\langle x'(t),\ 0,\ \frac{\partial f}{\partial x}\, x'(t) \right\rangle +$$

$$\left\langle 0,\ y'(t),\ \frac{\partial f}{\partial y}\, y'(t) \right\rangle$$

LIES ON THE TANGENT PLANE!

E.G.

INFLATABLE BASIN



$r(t)$
$h(t)$ CHANGE WITH TIME

AT $t = 3$ MIN $r = 1.5$ m, $h = 60$ cm

AND $\dfrac{dr}{dt} = 10 \dfrac{\text{cm}}{\text{min}}$ $\quad \dfrac{dh}{dt} = 5 \dfrac{\text{cm}}{\text{min}}$

HOW FAST IS VOLUME CHANGING?

$$V(r, h) = \pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$\frac{\partial V}{\partial r} = 2\pi r h \qquad \frac{\partial V}{\partial h} = \pi r^2$$

AT $T = 3$, IN m AND MINs

$$\left.\frac{dr}{dT}\right|_{T=3} = .1 \ \frac{m}{\min} \qquad \frac{dh}{dT} = .05 \ \frac{m}{\min}$$

$$\frac{dV}{dT} = \left.\frac{\partial V}{\partial r}\right|_{(1.5,.6)} \cdot \left.\frac{dr}{dT}\right|_{T=3} + \left.\frac{\partial V}{\partial h}\right|_{(1.5,.6)} \cdot \left.\frac{dh}{dT}\right|_{T=3}$$

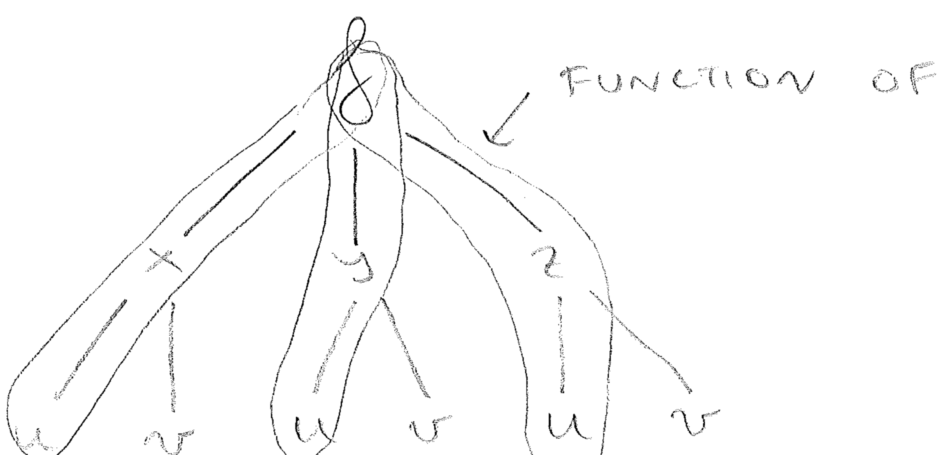
$$= \left(2\pi\cdot(1.5)(.6)(.1) + \pi(1.5)^2(.05)\right) \frac{m^3}{\min}$$

MORE GENERAL SITUATION:

$x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$

ARE PLUGGED INTO $f(x,y,z)$,

MAKING IT A FUNCTION OF u, v



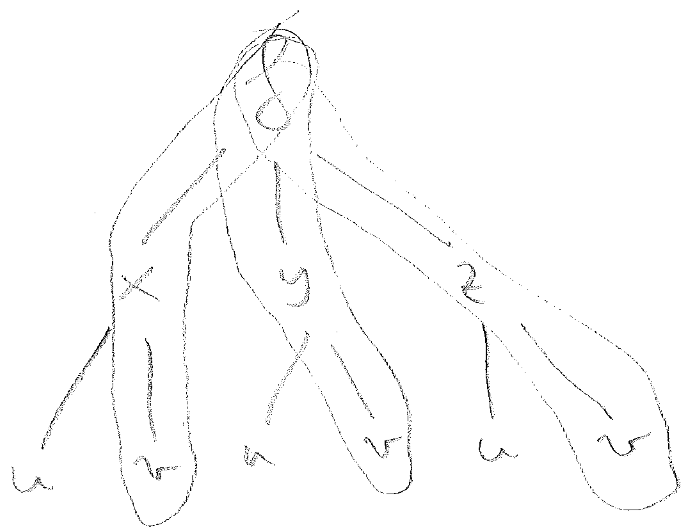
COLLECT BRANCHES ENDING IN

u:

$$\frac{\partial}{\partial u} f(x(u,v), y(u,v), z(u,v)) =$$

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u}$$

COLLECT BRANCHES ENDING IN v :



$$\frac{\partial}{\partial v} f(x(u,v), y(u,v), z(u,v)) =$$

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial v}$$

SAME THING FOR X, Y, Z

FUNCTIONS OF 3 VAR.

IN VECTOR NOTATION:

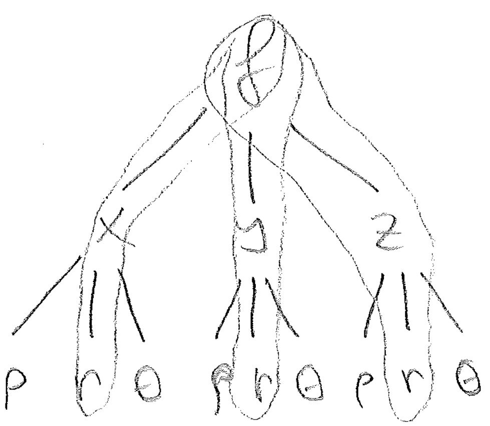
$$\frac{\partial f}{\partial u} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$$

THIS WORKS IN ANY NUMBER OF VAR!

E.G. $f(x, y, z) = x^2 + yz$

$x = \rho r \cos\theta \quad y = \rho r \sin\theta \quad z = \rho \cdot r$

FIND $\frac{\partial f}{\partial r}$ AT $(\rho_0, r_0, \theta_0) = (2, 3, \frac{\pi}{2})$



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

COMPUTE:

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = z \quad \frac{\partial f}{\partial z} = y$$

$$\frac{\partial x}{\partial r} = \rho\cos\theta \quad \frac{\partial y}{\partial r} = \rho\sin\theta \quad \frac{\partial z}{\partial r} = \rho$$

EVALUATE:

$$x\left(2,3,\tfrac{\pi}{2}\right) = 2\cdot 3\cdot \cos\tfrac{\pi}{2} = 0$$

$$y\left(2,3,\tfrac{\pi}{2}\right) = 2\cdot 3 \sin\tfrac{\pi}{2} = 6$$

$$z\left(2,3,\tfrac{\pi}{2}\right) = 2\cdot 3 = 6$$

SO

$$\frac{\partial f}{\partial r}\left(2,3,\tfrac{\pi}{2}\right) = \left.\frac{\partial f}{\partial x}\right|_{(0,6,6)} \left.\frac{\partial x}{\partial r}\right|_{(2,3,\frac{\pi}{2})}$$

NOT $\left(2,3,\tfrac{\pi}{2}\right)$!

$$+ \left.\frac{\partial f}{\partial y}\right|_{(0,6,6)} \left.\frac{\partial y}{\partial r}\right|_{(2,3,\frac{\pi}{2})} + \left.\frac{\partial f}{\partial z}\right|_{(0,6,6)} \left.\frac{\partial z}{\partial r}\right|_{(2,3,\frac{\pi}{2})}$$

$$= 2\cdot 0\cdot\left(2\cos\tfrac{\pi}{2}\right) + 6\cdot 2\sin\tfrac{\pi}{2} + 6\cdot 2 = 24$$

NOTE:

DO NOT GET CONFUSED BY THE FACT THAT ρ, r, θ ARE THE SAME NUMBER OF VAR. AS x, y, z; <u>YOU CANNOT PLUG ρ, r, θ DIRECTLY INTO f</u> OR ITS PARTIAL DERIVATIVES! YOU NEED TO COMPUTE $x(\rho, r, \theta)$, $y(\ldots)$ $z(\ldots)$.

ACTUALLY, IF f IS INTENDED TO "EAT" x, y, z AND x, y, z ARE FUNCTIONS OF ρ, r, θ, $f(\rho, r, \theta)$ IS AN <u>ACCEPTABLE SHORT NOTATION</u> FOR

$$f(x(\rho, r, \theta), y(\rho, r, \theta), z(\rho, r, \theta))\ !$$

CHAIN RULE AND IMPLICIT DIFF

SUPPOSE WE HAVE A SURFACE DESCRIBED BY $F(x, y, z) = 0$, AND LOCALLY WE HAVE $z = z(x, y)$.

THEN WE CAN USE THE CHAIN RULE IN A SMART WAY:

X AND Y ARE NATURALLY FUNCTIONS OF x, y ... $X(x, y) = x$, $y(x, y) = y$. SO ON OUR SURFACE

$$F(x(x, y), y(x, y), z(x, y)) = 0$$

THEN BY CHAIN RULE

$$F_x \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{=0} + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

SAME

$$F_x \cdot \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

E.G.

$$F(x, y, z) = \sin(xz + y)$$

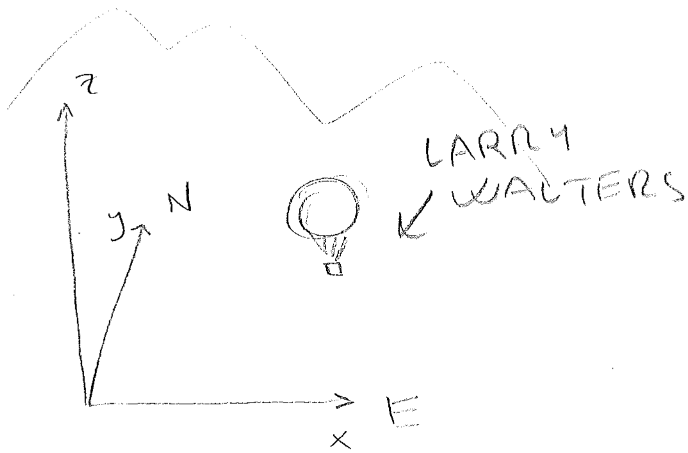
ON $F(x, y, z) = 0$ WE HAVE

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-z\cancel{\cos(xz+y)}}{x\cancel{\cos(xz+y)}} = \frac{-z}{x}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-\cos(xz+y)}{x\cos(xz+y)} = \frac{-1}{x}$$

WE NO LONGER NEED TO SOLVE AN EQUATION!

E.G.



HOT AIR BALLOON

AFTER 30 MINUTES THE BALLOON IS AT $(10, 5, 3)$.

UNITS IN HUNDREDS OF METERS.

AT THIS POINT

$T(x, y, z)$ = AIR TEMP AT (x, y, z)

$$\left.\frac{\partial T}{\partial x}\right|_{(10,5,3)} = 0.2^\circ / 100\,m$$

$$\left.\frac{\partial T}{\partial y}\right|_{(10,5,3)} = -0.3^\circ / 100\,m$$

$$\left.\frac{\partial T}{\partial z}\right|_{(10,5,3)} = -1^\circ / 100\,m$$

$\bar{V}$ = VELOCITY OF BALLOON =

$\langle 20\,m/min,\ 10\,m/min,\ 1\,m/min \rangle$ AT THIS POINT.

HOW FAST IS THE TEMPERATURE CHANGING FOR THE PASSENGER?

$$\bar{V} = \langle x'(T), y'(T), z'(T) \rangle$$

WE WANT $\left.\frac{d}{dT} T(x(T), y(T), z(T))\right|_{T=30}$

30 = MOMENT WHEN BALLOON IS AT (10, 5, 3).

CHAIN RULE $\frac{d}{dT} T = \langle T_x, T_y, T_z \rangle \cdot \langle x', y', z' \rangle$

$$\left.\frac{d}{dT} T\right|_{T=30} = \left.\frac{\partial T}{\partial x}\right|_{(10,5,3)} x'(30) + \left.\frac{\partial T}{\partial y}\right|_{(10,5,3)} y'(30)$$

$$+ \left.\frac{\partial T}{\partial z}\right|_{(10,5,3)} z'(30) = 0.2^\circ/100\,m \cdot 0.2 \left(100\,m/min\right)$$

$$+ \left(-6.3^\circ/100\,m\right)\left(0.1 \cdot 100\,m/min\right) + \left(-1^\circ/100\,m\right)\left(0.01 \cdot 100\,m/min\right)$$

$$= (0.04 - 0.03 - 0.01)^\circ/min = 0^\circ/min$$

TEMPERATURE IS CONSTANT!

WE SHOULD GIVE A NAME TO THE VECTOR $\langle T_x, T_y, T_z \rangle$.