$$\begin{cases} 3 + 12\tau = -21 \\ 1 - 7\tau = 15 \\ 1 + 11\tau = -21 \end{cases} \sim \begin{cases} 3 - 24 = -21 \\ 1 + 14 = 15 \\ \tau = -2 \end{cases} \checkmark$$

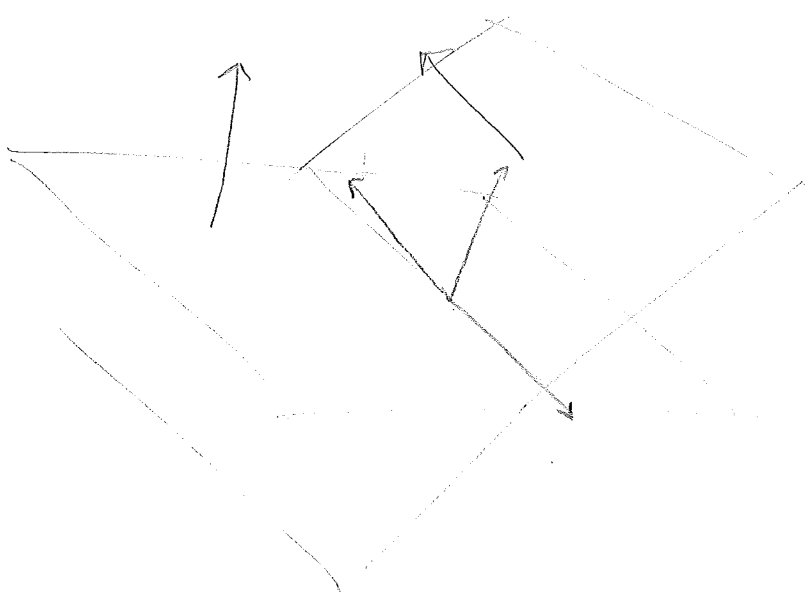
So $\ell_1 = \ell_2$!

INTERSECTING PLANES

TWO PLANES H_1, H_2 ARE PARALLEL IF THEY SHARE THE SAME PERPENDICULAR DIRECTION. IF $H_1 \parallel H_2$ THEN THEY EITHER ARE THE SAME OR NEVER MEET. IF $H_1 \not\parallel H_2$ THEN THEY MEET IN A LINE.

TO FIND THIS LINE

$$H_1 : A_1 x + B_1 y + C_1 z = D_1$$
$$H_2 : A_2 x + B_2 y + C_2 z = D_2$$



DIRECTION:

EITHER FIND A SOLUTION TO

$$\begin{cases} A_1 x + B_1 y + C_1 z = 0 \\ A_2 x + B_2 y + C_2 z = 0 \end{cases}$$

OR JUST TAKE

$$\langle A_1, B_1, C_1 \rangle \times \langle A_2, B_2, C_2 \rangle$$

WHY? $\vec{v}$ LIES ON $H_1 \iff \vec{v} \perp \langle A_1, B_1, C_1 \rangle$

$\vec{v}$ " " $H_2 \iff \vec{v} \perp \langle A_2, B_2, C_2 \rangle$

$\langle A_1, B_1, C_1 \rangle \times \langle A_2, B_2, C_2 \rangle$ IS $\perp$ TO BOTH!

POINT: FIND A RANDOM SOLUTION TO

$$\begin{cases} A_1 x + B_1 y + C_1 z = D_1 \\ A_2 x + B_2 y + C_2 z = D_2 \end{cases}$$

E.G.: H_1: $x + z = 5$

H_2: $y + 2z = -3$

ORTH VECTORS: $\langle 1, 0, 1 \rangle$, $\langle 0, 1, 2 \rangle$

SO DIRECTION OF $\ell = H_1 \cap H_2$ IS

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} =$$

$$\langle -1, -2, 1 \rangle$$

POINT:

$$\begin{cases} x + z = 5 \\ y + 2z = -3 \end{cases} \qquad \text{PICK } x = 0 \quad \text{SO } z = 5$$

$$y + 10 = -3 \qquad y = -13$$

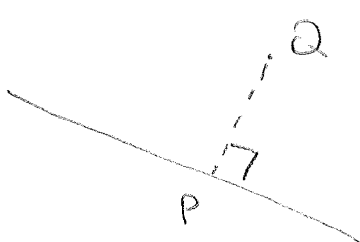
$$P = (0, 5, -13)$$

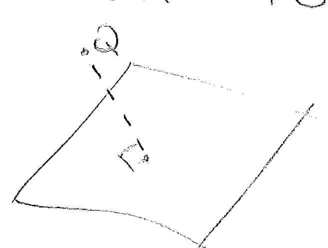
$$\vec{\ell}(t) = \langle 0, 5, -13 \rangle + t \langle -1, -2, 1 \rangle$$

DISTANCE FROM A LINE / PLANE

THE DISTANCE FROM A POINT Q TO A REGION R IS THE MINIMUM LENGTH OF $\overline{PQ}$ WHERE P BELONGS TO R.

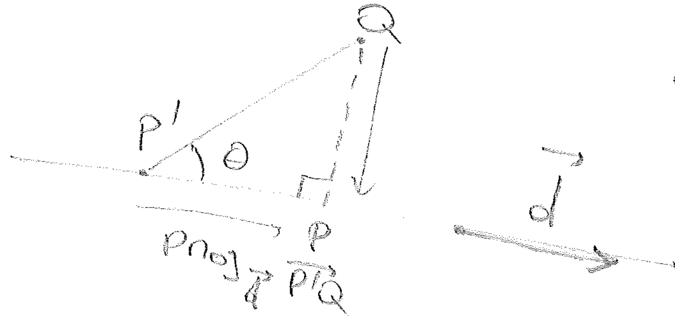
IF R IS A LINE OR A PLANE, THE MINIMUM DISTANCE IS OBTAINED BY FINDING A POINT P SUCH THAT $\overrightarrow{PQ}$ IS PERPENDICULAR TO R.





HOW DO WE DO IT?

LINE:



P' RANDOM

$\|\overrightarrow{PQ}\|$

$= \sin\theta \, \|\overrightarrow{P'Q}\|$

NOW, $\|\vec d \times \overrightarrow{P'Q}\| = \sin\theta \, \|\vec d\| \, \|\overrightarrow{P'Q}\|$

SO DISTANCE $= \|\overrightarrow{PQ}\| = \dfrac{\|\vec d \times \overrightarrow{P'Q}\|}{\|\vec d\|}$

SAME IDEA:

$\|\overrightarrow{PQ}\| = \|\overrightarrow{P'Q} - \text{Proj}_{\vec d}\,\overrightarrow{P'Q}\|$

E.G.

$Q = (-1, 0, 3)$ $\quad \vec{\ell}(t) = \langle 1, -1, 1\rangle + t\langle 2, 3, 1\rangle$

RANDOM POINT: $P' = (1, -1, 1)$

$\overrightarrow{P'Q} = \langle -2, 1, 2\rangle$, $\quad \vec d = \langle 2, 3, 1\rangle$

METHOD 1: $\vec c = \overrightarrow{P'Q} \times \vec d = \begin{vmatrix} \vec i & \vec j & \vec k \\ -2 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = \vec i \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - \vec j \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$
$+ \vec k \begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} = \langle -5, 6, -8\rangle$

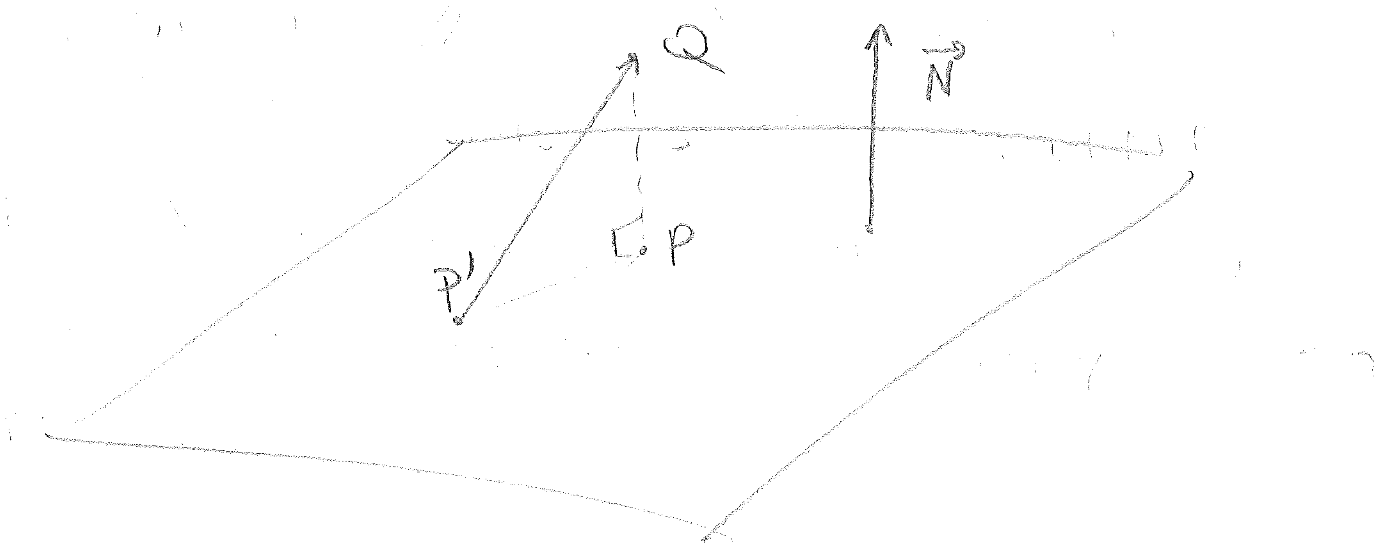
$\dfrac{\|\vec c\|}{\|\vec d\|} = \dfrac{\sqrt{125}}{\sqrt{14}} = \dfrac{5\sqrt5}{\sqrt{14}}$

METHOD 2: $\text{Proj}_{\vec d}\,\overrightarrow{P'Q} = \dfrac{1}{14}\langle 2, 3, 1\rangle$

$\vec{e} = \overrightarrow{P'Q} - \mathrm{Proj}_{\vec{d}} \overrightarrow{P'Q} = \frac{1}{14} \langle -30, 11, 27 \rangle$

$$\|\vec{e}\| = \frac{\sqrt{1750}}{14} = \frac{\sqrt{125 \cdot 14}}{14} = \frac{5\sqrt{5}}{\sqrt{14}}$$

DISTANCE FROM A POINT TO A PLANE



TO FIND $\overrightarrow{QP}$, JUST START FROM A RANDOM POINT P', THEN TAKE THE PROJECTION OF $\overrightarrow{P'Q}$ ALONG THE ORTHOGONAL DIRECTION $\vec{N}$

$$\text{DISTANCE} = \|\mathrm{Proj}_{\vec{N}} \overrightarrow{P'Q}\| = \frac{|\overrightarrow{P'Q} \cdot \vec{N}|}{\|\vec{N}\|}$$

E.G. $Q = (2,2,3)$, $H: x - y + 2z = 3$

$\vec{N} = \langle 1, -1, 2 \rangle$ $\quad P' = (1,0,1)$ $\quad \overrightarrow{P'Q} = \langle 1, 2, 2 \rangle$

$$\frac{\overrightarrow{P'Q} \cdot \vec{N}}{\|\vec{N}\|} = \frac{3}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{2}}$$