EQUATIONS OF PLANES IN $\mathbb{R}^3$

GENERAL (IMPLICIT) EQUATION

$$Ax + By + Cz = D \sim \langle A, B, C\rangle \cdot \langle x, y, z\rangle = D$$

THE VECTOR $\langle A, B, C\rangle$ IS <u>PERPENDICULAR</u> TO THE PLANE.

WHY: P, Q ON PLANE $P = (x_0, y_0, z_0)$
$Q = (x_1, y_1, z_1)$ $\quad \vec{PQ} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

$$\langle A, B, C\rangle \cdot \vec{PQ} = A(x_1 - x_0) + B(y_1 - y_0) + C(z_1 - z_0)$$
$$= (Ax_1 + By_1 + Cz_1) - (Ax_0 + By_0 + Cz_0) =$$
$$C - C = 0.$$

IF YOU KNOW A POINT $P = (x_0, y_0, z_0)$ ON THE PLANE, YOU CAN WRITE THE EQUATION IN THE <u>STANDARD FORM</u>

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

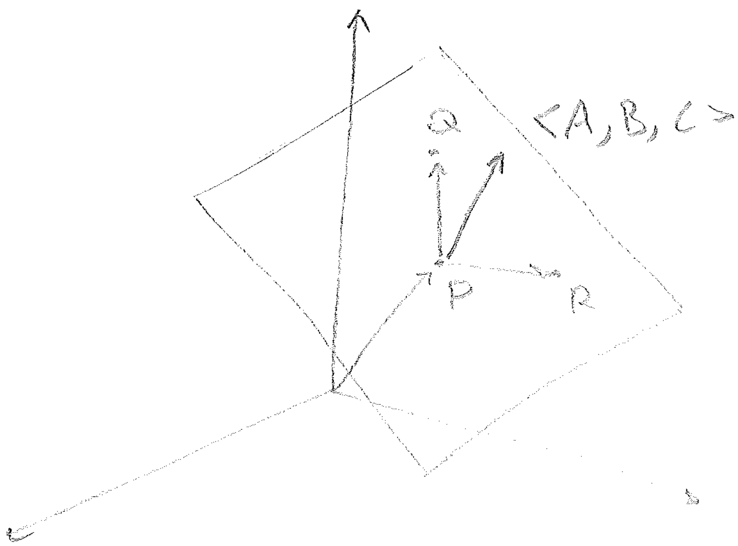
AND $D = \langle A, B, C\rangle \vec{OP} = Ax_0 + By_0 + Cz_0$

GIVEN THREE PTS P, Q, R ON THE PLANE YOU CAN WRITE A

PARAMETRIC EQUATION

$$\langle x, y, z\rangle = \underset{\langle x_0, y_0, z_0\rangle}{\underset{\shortparallel}{\overrightarrow{OP}}} + T_0 \overset{\langle a_0, b_0, c_0\rangle}{\overset{\shortparallel}{\overrightarrow{PQ}}} + T_1 \overrightarrow{PR} = \langle a_1, b_1, c_1\rangle$$

$$x = x_0 + T_0 a_0 + T_1 a_1$$
$$y = y_0 + T_0 b_0 + T_1 b_1$$
$$z = z_0 + T_0 c_0 + T_1 c_1$$



AND OBTAIN THE PREVIOUS EQUATIONS BY EITHER

1) FINDING A SOLUTION TO
$$\begin{cases} \langle A, B, C\rangle \cdot \overrightarrow{PQ} = 0 \\ \langle A, B, C\rangle \cdot \overrightarrow{PR} = 0 \\ \langle A, B, C\rangle \cdot \overrightarrow{OP} = D \end{cases}$$

2) COMPUTING
$$\begin{cases} \langle A, B, C\rangle = \overrightarrow{PQ} \times \overrightarrow{PR} \\ \langle A, B, C\rangle \cdot \overrightarrow{OP} = D \end{cases}$$

E.G. FIND THE GENERAL, STANDARD, PARAMETRIC EQUATIONS OF THE PLANE THROUGH

$P=(1,0,1)$, $Q=(1,2,-1)$, $R=(0,1,2)$

PARAMETRIC: $\vec{OP}+t_0\vec{PQ}+t_1\vec{PR}=\langle x,y,z\rangle$

$\langle 1,0,1\rangle + t_0\langle 0,2,-2\rangle + t_1\langle -1,1,1\rangle = \langle x,y,z\rangle$

$x=1-t_1 \quad y=2t_0-t_1 \quad , \quad z=1-2t_0+t_1$

GENERAL:

1) $\begin{cases} 2B-2C=0 \\ -A+B+C=0 \end{cases} \sim \begin{cases} B=C \\ A=2B \end{cases}$

$A=2, B=1, C=1$ IS A SOL

$D=\langle 2,1,1\rangle\langle 1,0,1\rangle = 3$

$2x+y+z=3$

2) $\vec{PQ}\times\vec{QR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -2 \\ -1 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & -2 \\ -1 & 1 \end{vmatrix}$

$+\vec{k}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = \langle 4,2,2\rangle \quad D=\langle 4,2,2\rangle\langle 1,0,1\rangle$

$=6$

STANDARD: $(x_0,y_0,z_0)=(1,0,1)$

$2(x-1)+(y-1)+(z-1)=0$

EQUATIONS OF LINES IN $\mathbb{R}^3$

WE CAN DEFINE A LINE EXPLICITLY WITH:

TWO POINTS Q, P $\quad \overrightarrow{\ell(t)} = \overrightarrow{OP} + t\,\overrightarrow{PQ}$

POINT + DIRECTION $Q, \vec{d}$ $\quad \overrightarrow{\ell(t)} = \overrightarrow{OQ} + t\,\vec{d}$

THE ABOVE ARE <u>VECTOR PARAMETRIC EQUATIONS</u>, EQUIVALENT TO THE <u>PARAMETRIC EQUATION</u>

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt,$$
$$z(t) = z_0 + ct$$

$\overrightarrow{\ell(t)}$ IS AN EXAMPLE OF A VECTOR VALUED FUNCTION. IF $a, b, c \neq 0$ ONE CAN "SOLVE FOR t" TO OBTAIN $t = \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$

THE EQUATIONS

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

ARE CALLED THE <u>SYMMETRIC EQUATIONS</u> FOR THE LINE.

E.G. FIND THE VECTOR, PARAMETRIC AND SYMMETRIC EQUATIONS OF THE LINE ℓ THROUGH $P=(1,3,3)$, $Q=(3,1,2)$

- $\vec{\ell}(t) = \vec{OP} + t\,\vec{PQ} = \langle 1,3,3\rangle + t\langle 2,-2,-1\rangle$
- $x(t) = 1+2t$, $y(t) = 3-2t$, $z(t) = 3-t$
- $\dfrac{x-1}{2} = -\dfrac{y-3}{2} = -\dfrac{z-3}{1}$

CHECK: $\dfrac{3-1}{2} = -\dfrac{1-3}{2} = -\dfrac{2-3}{1} = 1$ ✓

E.G. DOES THE LINE THROUGH $Q=(0,1,0)$ WITH DIRECTION $\vec{d} = \langle 3,0,2\rangle$ ADMIT SYMMETRIC EQ.?

NO: $\vec{\ell}(t) = \vec{OQ} + t\vec{d} = \langle 0,1,0\rangle + t\langle 3,0,2\rangle$

- $x(t) = 3t$, $y(t) = 1$, $z(t) = 2t$
- NO SYM EQ AS $b = 0$.

PARALLEL, INTERSECTING, SKEW LINES

ℓ_1, ℓ_2 LINES. ℓ_1 AND ℓ_2 ARE PARALLEL ($\parallel$) IF THEY HAVE THE SAME DIRECTION.

IF ADDITIONALLY THEY INTERSECT, THEY ARE THE SAME LINE.

IF $\ell_1 \not\parallel \ell_2$, THEY CAN BE

- SKEW IF THEY DO NOT INTERSECT
- INTERSECTING IF THEY SHARE A POINT.

	$\parallel$	$\not\parallel$
MEET ?	SAME	INCIDENT
DON'T MEET	PARALL.	SKEW

<u>E.G.</u>: ℓ_1: $x = 1+3t$, $y = 2-t$, $z = t$

ℓ_2: $x = -2+4s$ $y = 3+s$ $z = 5+2s$

DIRECTIONS: $\vec{d}_1 = \langle 3, -1, 1 \rangle$

$\vec{d}_2 = \langle 4, 1, 2 \rangle$ ARE NOT MULT OF EACH OTHER, SO $\ell_1 \not\parallel \ell_2$

<u>MEET ?</u>:

$$\begin{cases} 1+3t = -2+4s \\ 2-t = 3+s \\ t = 5+2s \end{cases} \xrightarrow{\text{SUB}} \begin{cases} 1+3t = -2+4s \\ s = -2 \\ t = 5+2s \end{cases} \text{SUB}$$

$$\sim \begin{cases} 1+3t = -2+4s \\ s = -2 \\ t = 1 \end{cases} \sim \begin{cases} 4 = -10 \end{cases} \leftarrow \text{ABSURD}$$

SO ℓ_1, ℓ_2 ARE SKEW

<u>E.G.</u> $\ell_1 : \langle x, y, z \rangle = \langle 3, 1, 1 \rangle + t\langle 12, -7, 11 \rangle$

$\ell_2 : \langle x, y, z \rangle = \langle -21, 15, -21 \rangle + s\langle -9, \frac{21}{4}, -\frac{33}{4} \rangle$

DIRECTIONS: $\vec{d}_1 = \langle 12, -7, 11 \rangle$

$\vec{d}_2 = \langle -9, \frac{21}{4}, -\frac{33}{4} \rangle$

$\frac{12}{-9} = -\frac{4}{3}$ SO IF THEY ARE MULTIPLES WE MUST HAVE

$\vec{d}_1 = -\frac{4}{3}\vec{d}_2$; CHECK

$-\frac{4}{3}\vec{d}_2 = \langle 12, -7, 11 \rangle = \vec{d}_1$ SO

$\ell_1 \parallel \ell_2$. ARE THEY EQUAL?

NOTE: IF THEY MEET, THEY ARE EQUAL, SO WE CAN CHECK <u>ANY</u> POINT

$$\begin{cases} 3 + 12\tau = -21 \\ 1 - 7\tau = 15 \\ 1 + 11\tau = -21 \end{cases} \sim \begin{cases} 3 - 24 = -21 \\ 1 + 14 = 15 \\ \tau = -2 \end{cases} \checkmark$$

So $\ell_1 = \ell_2$!

INTERSECTING PLANES

TWO PLANES H_1, H_2 ARE PARALLEL IF THEY SHARE THE SAME PERPENDICULAR DIRECTION. IF $H_1 \parallel H_2$ THEN ~~THEY~~ EITHER ARE THE SAME OR NEVER MEET. IF $H_1 \not\parallel H_2$ THEN THEY MEET IN A LINE.

TO FIND THIS LINE

$$H_1 : A_1 x + B_1 y + C_1 z = D_1$$
$$H_2 : A_2 x + B_2 y + C_2 z = D_2$$