Worksheet 2: lines and planes

1. Find an equation of the plane containing the points $P = (3, 0, 0)$, $Q = (0, 2, 0)$ and $R = (1, 0, 0)$.

2. Find an equation for the line of intersection of the planes with equations $x - y + 2z = 0$ and $3y + z = 0$.

3. Let the lines l_1 and l_2 be given by the parametric equations

$$\vec{l}_1(t) = t\vec{i} + (1 - 2t)\vec{j} + (2 + 3t)\vec{k},$$

$$\vec{l}_2(s) = (3 - 4s)\vec{i} + (2 + 3s)\vec{j} + (1 - 2s)\vec{k}.$$

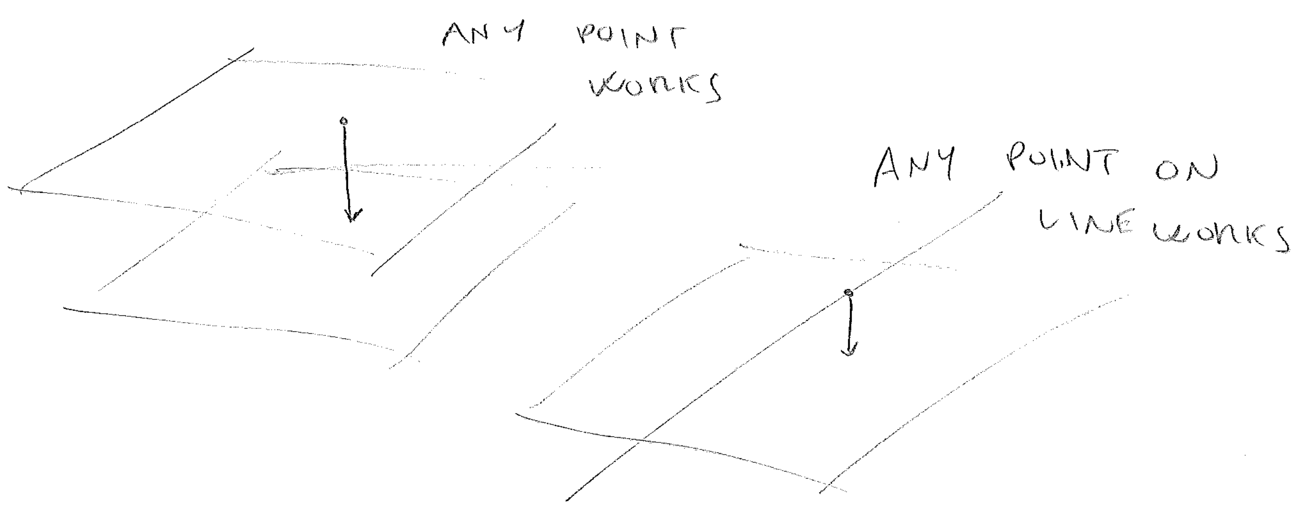
Do these lines intersect?

DISTANCE BETWEEN REGIONS

THE DISTANCE BETWEEN TWO REGIONS R_1, R_2 IS THE MINIMUM LENGTH OF A VECTOR $\vec{PQ}$ WHERE P BELONGS TO R_1 AND Q TO R_2

BORING CASES: PLANE TO PLANE, LINE TO PLANE

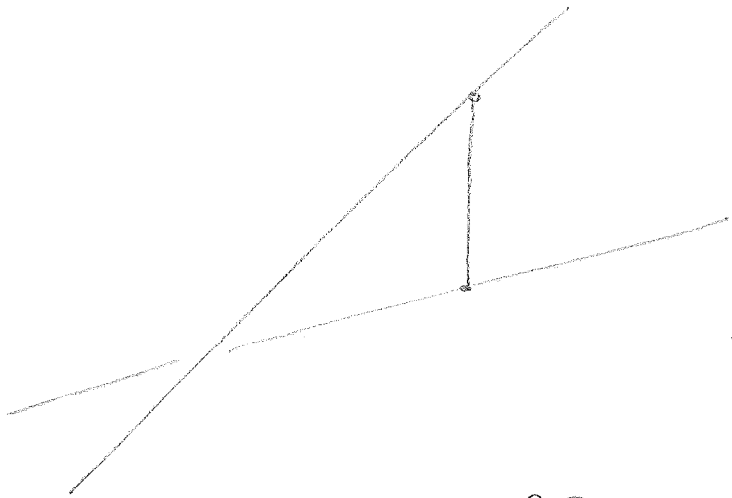
- IN BOTH, EITHER THE TWO ARE // OR THE DISTANCE IS 0 (THEY SHARE A POINT).
- IF THEY ARE PARALLEL, JUST PICK A POINT ON ONE (THE LINE IN THE SECOND CASE). THE DISTANCE IS THE DISTANCE TO THE POINT.



MORE BORING CASES: INCIDENT OR PARALLEL LINES

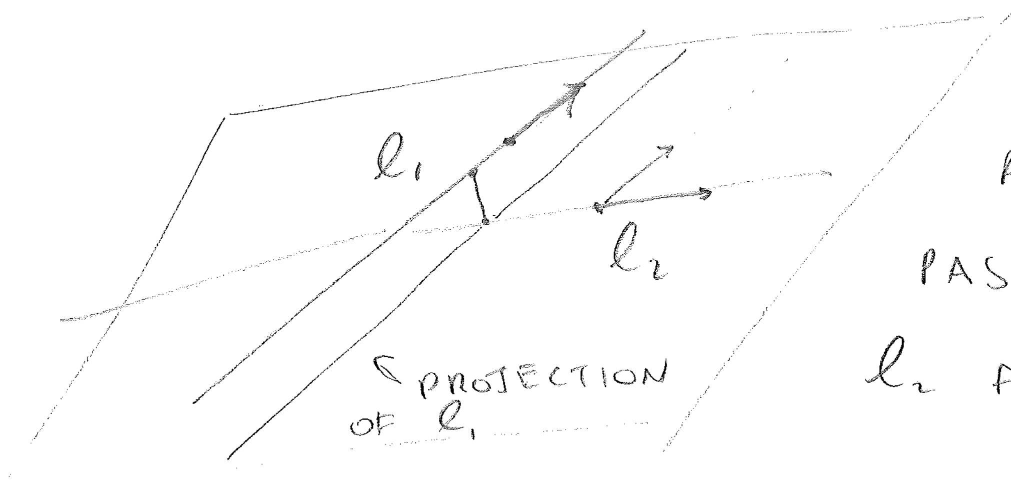
- AS ABOVE

INTERESTING CASE: SKEW LINES



HOW DO WE FIND TWO POINTS P, Q TO MINIMIZE DISTANCE? LOOKS HARD!

IDEA: REDUCE TO BORING CASE



WE CONSTRUCT A <u>PLANE</u> H PASSING THROUGH ℓ_2 AND // TO ℓ_1

THEN THE DISTANCE FROM ℓ_1 TO ℓ_2 IS THE DISTANCE FROM ℓ_1 TO H.

<u>HOW?</u> $\vec{\ell_1}(t) = \vec{OP} + t\,\vec{d_1}$

$\vec{\ell_2}(t) = \vec{OQ} + s\,\vec{d_2}$

THEN H IS GIVEN BY

$\langle x, y, z \rangle = \vec{OQ} + t\,\vec{d_1} + s\,\vec{d_2}$

ITS NORMAL DIRECTION IS GIVEN BY $\vec{N} = \vec{d}_2 \times \vec{d}_2$ AND THE IMPLICIT EQUATION IS

$$\langle x, y, z \rangle \cdot \vec{N} = \vec{OQ} \cdot \vec{N}$$

<u>E.G.</u> $\vec{\ell_1}(t) = \langle 0, 1, 2 \rangle + t \langle 1, -2, 3 \rangle$

$\vec{\ell_2}(s) = \langle 3, 2, 1 \rangle + s \langle -4, 3, -2 \rangle$

$\vec{d}_1 = \langle 1, -2, 3 \rangle \quad \vec{d}_2 = \langle -4, 3, -2 \rangle$

$$\vec{N} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 3 \\ -4 & 3 & -2 \end{vmatrix} = \langle -5, -10, -5 \rangle$$

SO $H: -5x - 10y - 5z = \langle 3, 2, 1 \rangle \cdot \vec{N} = -40$

SAME AS $x + 2y + z = 8$, $\vec{N} = \langle 1, 2, 1 \rangle$

POINT P IN ℓ_1 $P = (0, 1, 2)$, Q IN H

$Q = (3, 2, 1)$, $\vec{PQ} = \langle -3, -1, 1 \rangle$

DISTANCE IS $\dfrac{|\vec{PQ} \cdot \vec{N}|}{\|\vec{N}\|} = \dfrac{4}{\sqrt{6}}$

<u>POINT TO PLANE</u>

QUADRIC SURFACES

A QUADRIC SURFACE IS THE SURFACE DEFINED BY A DEGREE TWO POLYNOMIAL EQUATION IN x, y, z.

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Gx+Hy+Iz+J=0$$

BY ROTATING OUR 3-D SPACE, WE CAN ALWAYS MAKE IT SO THE D, E, F COEFFICIENTS ARE 0, SO OUR EQUATION SIMPLIFIES TO

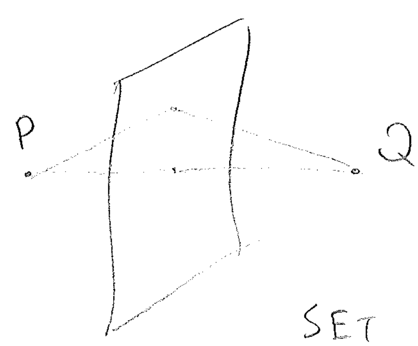
$$Ax^2+By^2+Cz^2+Gx+Hy+Iz+J=0$$

E.G. 1) $x^2+y^2+z^2=R^2$ SPHERE CENTER 0, RADIUS R

2) $\underbrace{x^2+2x}+y^2+z^2=R^2$

$(x+1)^2-1+y^2+z^2=R^2$ SPHERE CENTER

$(x+1)^2+y^2+z^2=R^2+1$ $(-1,0,0)$, RADIUS $\sqrt{R^2+1}$

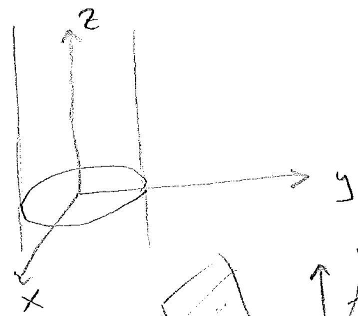
3) 

$(x-a)^2+(y-b)^2+(z-c)^2$
$=(x-d)^2+(y-e)^2+(z-f)^2$

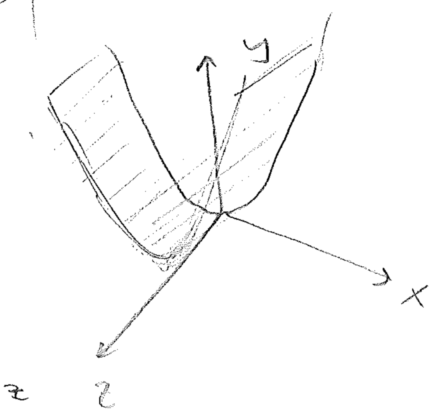
SET OF PTS EQUIDISTANT FROM P, Q; ALL SQUARE TERMS CANCEL OUT, WE GET A PLANE

4) IF ONE OF THE VARIABLES DOES NOT APPEAR, WE GET A CYLINDER

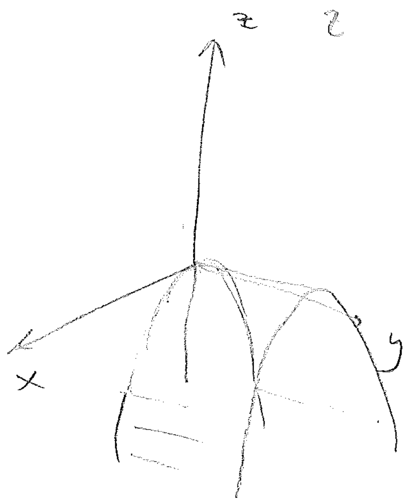
- $x^2 + y^2 = 1$



- $x^2 - y = 0$



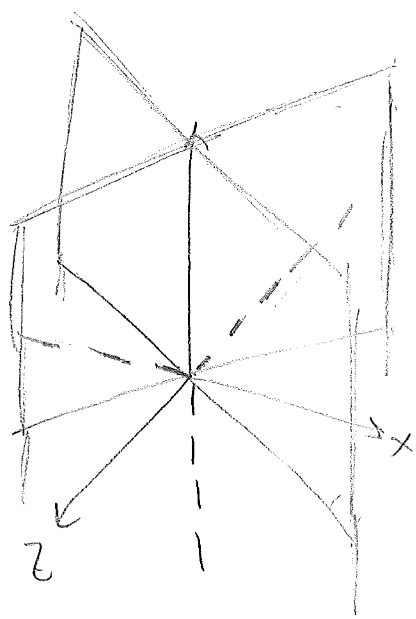
- $z^2 + x = 0$



- $z^2 - x^2 = 0$

$(z+x)(z-x) = 0$

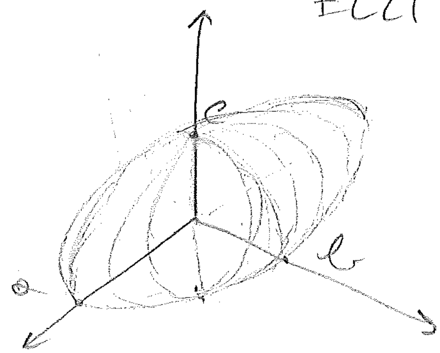
TWO LINES ON XZ PLANE



STANDARD EXAMPLES :

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

ELLIPSOID



2. $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

ELLIPTIC PARABOLOID

- HORIZONTAL TRACES ARE ELLIPSES

3. $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

HYPERBOLIC PARABOLOID

- HORIZONTAL TRACES ARE HYPERBOLAS

4. $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

CONE

5. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ HYPERBOLOID OF 1 SHEET

6. $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ HYPERBOLOID OF 2 SHEETS

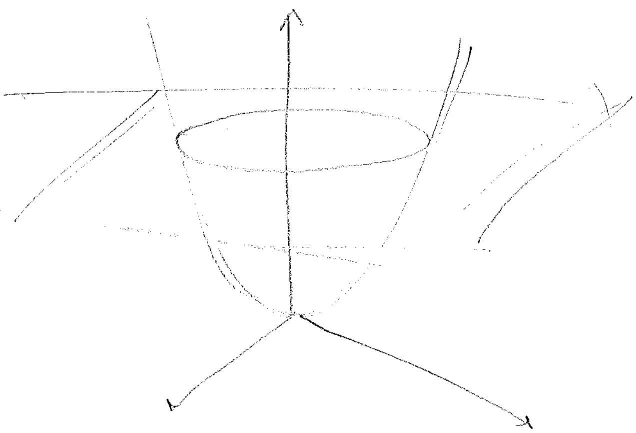
FACT : ANY QUADRIC SURFACE CAN BE REDUCED TO ONE OF THESE OR TO A CYLINDER

TRACES

A TRACE OF A QUADRATIC SURFACE IS ITS INTERSON WITH A PLANE PARALLEL TO

A COORDINATE PLANE (THAT IS, EVALUATE ONE OF THE VARIABLES). WE CAN TELL THE DIFFERENT TYPES OF SURFACES BY LOOKING AT TRACES.

E.C. IN CASE 2 SET $z = ck$ (SA[illegible])



$$ck = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

WE GET AN ELLIPSOID

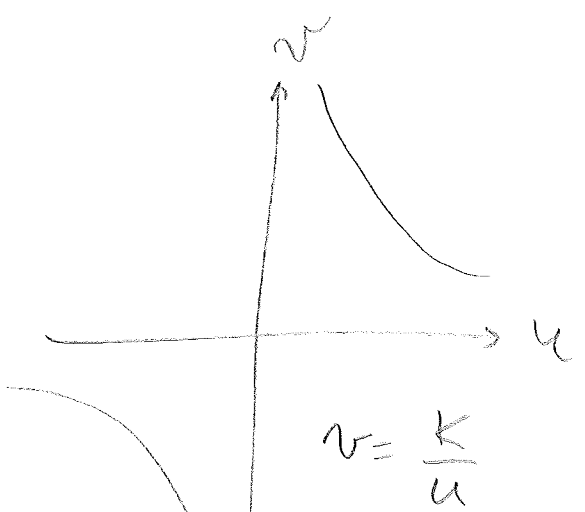
(EXAMPLE: IF $a = b = 1$ WE GET A CIRCLE)

IF $k = 0$ WE GET $(0, 0, 0)$

E.G. IN CASE 3 SET $z = ck$, $a = b = 1$

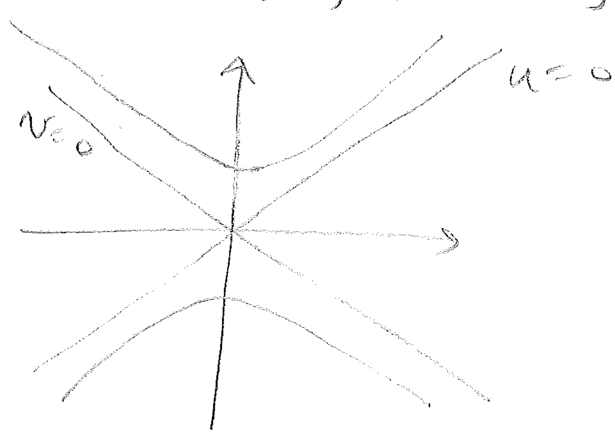
$$k = x^2 - y^2 = (x-y)(x+y)$$

$u = x - y$, $v = x + y$



$$v = \frac{k}{u}$$

HYPERBOLA



VERTICAL PLANES: $x = k$

2: $z = k^2 + y^2$ UPWARD PARABOLA

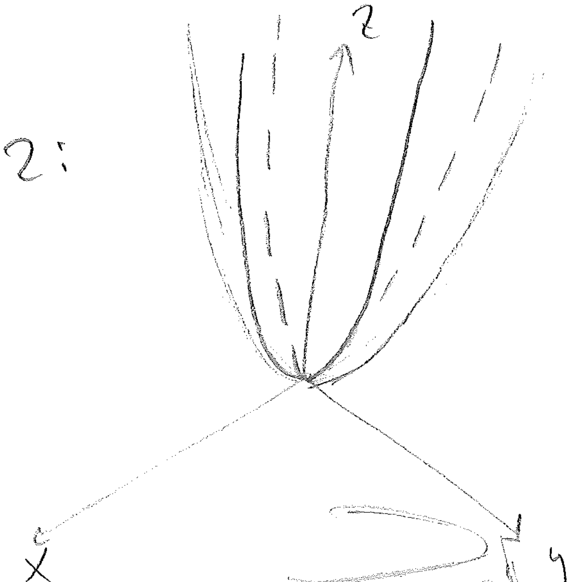
3: $z = k^2 - y^2$ DOWNWARD PARABOLA

$y = k$

2: $z = x^2 + k^2$ UPWARD PARABOLA

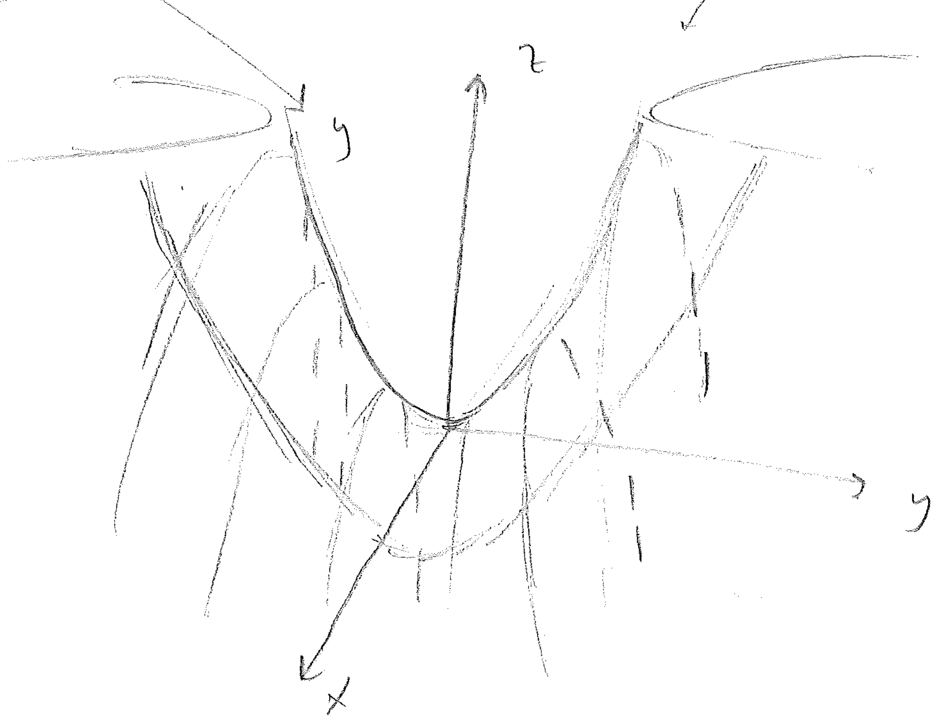
$z = x^2 - k^2$ UPWARD PARABOLA

2:



"HORSE SADDLE"

3:



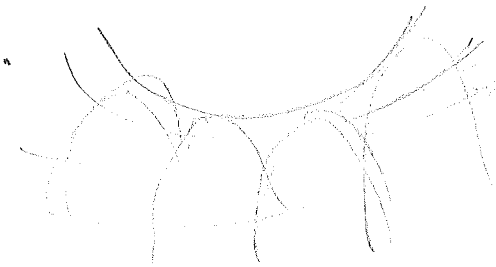
THE POINT:

WE WILL LATER CONSIDER LINEAR (TANGENT PLANE) AND QUADRATIC APPROXIMATION TO THE GRAPH OF A FUNCTION OF TWO VARIABLES. THESE WILL BE SURFACES LIKE $z = ax^2 + b y^2$ AND WILL LOOK LIKE

2. ("POSITIVELY CURVED")

SIGN a = SIGN b OR

3.



("NEGATIVELY CURVED")

SIGN $a \neq$ SIGN b