Worksheet 1: vectors and forces

1. What is the general form of a vector $\vec{v}$ on the plane that points upward and forms an angle of $\pi/3$ with the positive direction of the x-axis?

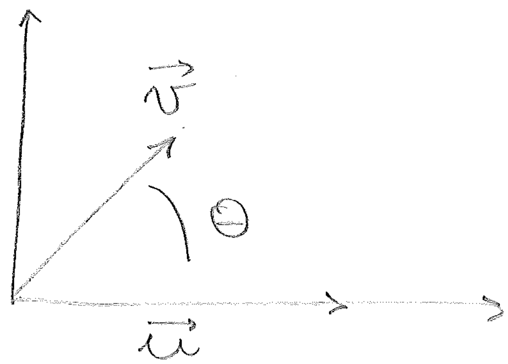
2. Find two vectors $\vec{v}_1$ and $\vec{v}_2$ on the plane such that $\vec{v}_1$ forms an angle of $\pi/6$ with the positive direction of the x-axis, $\vec{v}_2$ forms an angle of $3\pi/4$ with the positive direction of the x-axis, and $\vec{v}_1 + \vec{v}_2 = \langle 0, -10 \rangle$.

3. A small block of mass 1kg hangs on two chains such that one chain forms an angle of 45° and the other an angle of 30° with the horizontal. Find the forces that act on the block.

DOT PRODUCT AND ANGLES:

IN 2d: PICK $\vec{u} = a\vec{i} = \langle a, 0 \rangle$

$\vec{v} = b(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) = \langle b\cos\theta, b\sin\theta \rangle$



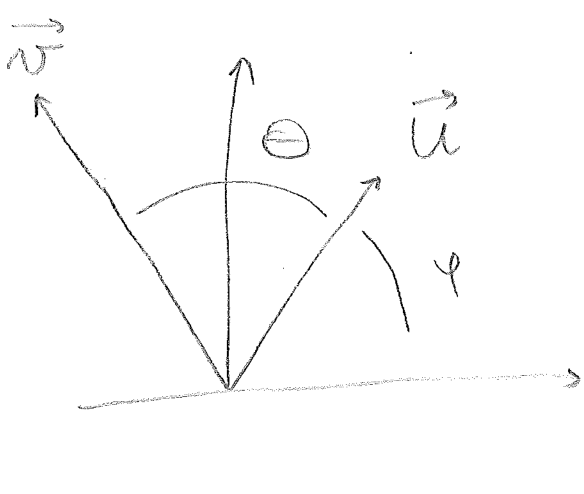
$\vec{u}\cdot\vec{v} = ab\cos\theta = \|\vec{u}\|\|\vec{v}\|\cos\theta$

IN GENERAL:

$\vec{u} = a\langle \cos\varphi, \sin\varphi \rangle$, $\vec{v} = b\langle \cos(\theta+\varphi), \sin(\theta+\varphi) \rangle$

$\vec{u}\cdot\vec{v} = a\cdot b\,(\underline{\cos^2\varphi}\cos\theta - \underbrace{\cos\theta\sin\varphi\sin\theta} +$
$\underline{\sin^2\varphi}\cos\theta + \underbrace{\sin\theta\sin\varphi\cos\theta}) =$

$a\cdot b\,(\cos\theta) = \|\vec{u}\|\|\vec{v}\|\cos\theta$!

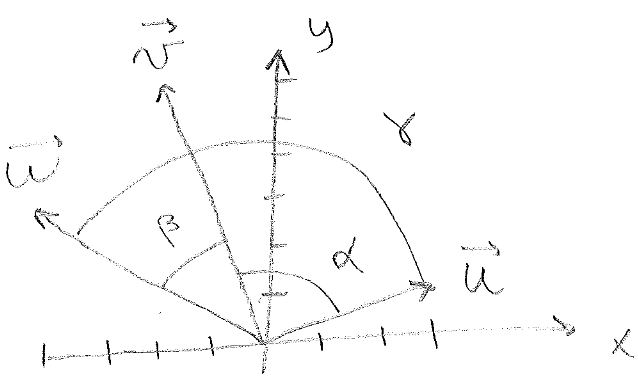


THM:

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ WHERE $0 \leq \theta \leq \pi$ IS THE ANGLE BETWEEN $\vec{u}$ AND $\vec{v}$ ON A PLANE CONTAINING BOTH.

THE MAIN USE OF THIS THM IS TO FIND ANGLES.

E.G.: FIND THE ANGLES BETWEEN $\vec{u} = \langle 3, 1 \rangle$, $\vec{v} = \langle -2, 6 \rangle$ AND $\vec{w} = \langle -4, 3 \rangle$



$\|\vec{u}\| = \sqrt{9+1} = \sqrt{10}$

$\|\vec{v}\| = \sqrt{40} = 2\sqrt{10}$

$\|\vec{w}\| = \sqrt{25} = 5$

$\vec{u} \cdot \vec{v} = -6 + 6 = 0 = 20 \cos\theta$

$\cos\theta = 0 \rightsquigarrow \theta = \arccos 0 \; \left(= \frac{\pi}{2}\right)$

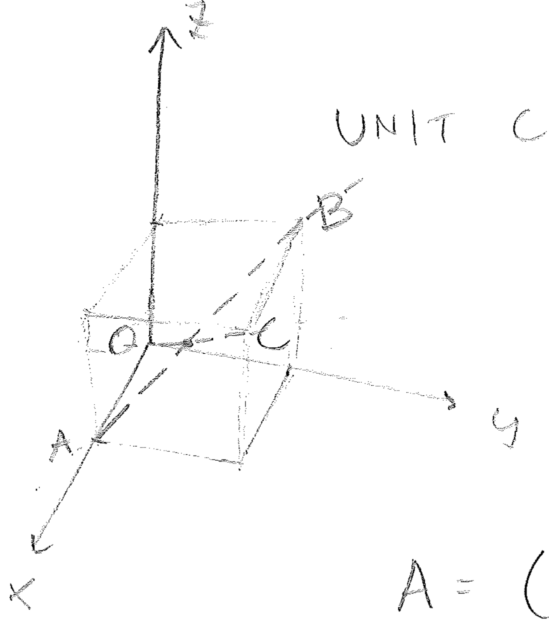
$\vec{v} \cdot \vec{w} = 8 + 18 = 26 = 10\sqrt{10} \cos\theta$

$\theta = \arccos\left(\frac{26}{10\sqrt{10}}\right) \quad (\approx 0.605)$

$\vec{u}\,\vec{w} = -12 + 3 = -9 = 5\sqrt{10} \cos\theta$

$\theta = \arccos\left(\frac{-9}{5\sqrt{10}}\right) \quad (\approx 2.176)$

E.G.: ANGLE BETWEEN LINES



UNIT CUBE. FIND THE ANGLE BETWEEN $\overline{AB}$ AND $\overline{OC}$

$A = (1,0,0)$ $\qquad \overrightarrow{AB} = \langle -1, 1, 1 \rangle$

$B = (0,1,1)$ $\qquad \overrightarrow{OC} = \langle 1, 1, 1 \rangle$

$C = (1,1,1)$

$$\overrightarrow{AB} \cdot \overrightarrow{OC} = -1+1+1 = 1$$

$$\|\overrightarrow{AB}\| = \|\overrightarrow{OC}\| = \sqrt{1+1+1} = \sqrt{3}$$

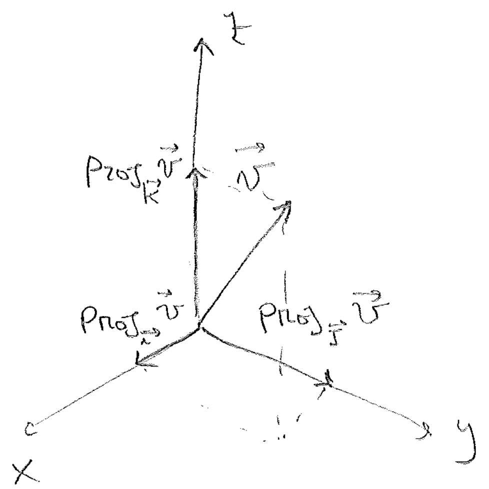
$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{OC}}{\|\overrightarrow{AB}\| \|\overrightarrow{OC}\|} = \frac{1}{3}$$

SO $\theta = \arccos \frac{1}{3}$; IN PARTICULAR THEY ARE NOT PERPENDICULAR

$$\left(\vec{v} \perp \vec{u} \iff \theta = \frac{\pi}{2} \iff \vec{v} \cdot \vec{u} = 0 \right)$$

PROJECTIONS AND COMPONENTS

$\vec{i}, \vec{j}, \vec{k}$ STANDARD VECTORS OF LENGTH 1



$\vec{v} = \langle a, b, c \rangle$

$\text{comp}_x \vec{v} = a$

$\text{comp}_y \vec{v} = b$

$\text{comp}_z \vec{v} = c$

COMPONENTS ARE NUMBERS

$\text{proj}_{\vec{i}} \vec{v} = a\vec{i}$

$\text{proj}_{\vec{j}} \vec{v} = b\vec{j}$

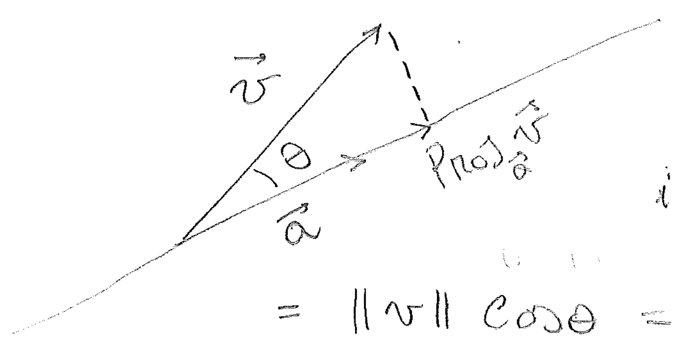
$\text{proj}_{\vec{k}} \vec{v} = c\vec{k}$

PROJECTIONS ARE VECTORS

NOTE: $a = \vec{v} \cdot \vec{i}$, $b = \vec{v} \cdot \vec{j}$, $c = \vec{v} \cdot \vec{k}$

NOW $\vec{a}, \vec{v}$ VECTORS

I WANT TO DEFINE A COMPONENT OF $\vec{v}$ ALONG $\vec{a}$ AND A PROJECTION OF $\vec{v}$ ALONG $\vec{a}$.



i) MAKE $\vec{a}$ A UNIT VEC

$$\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$$

ii) NOW $\text{Comp}_{\vec{a}} \vec{v}$

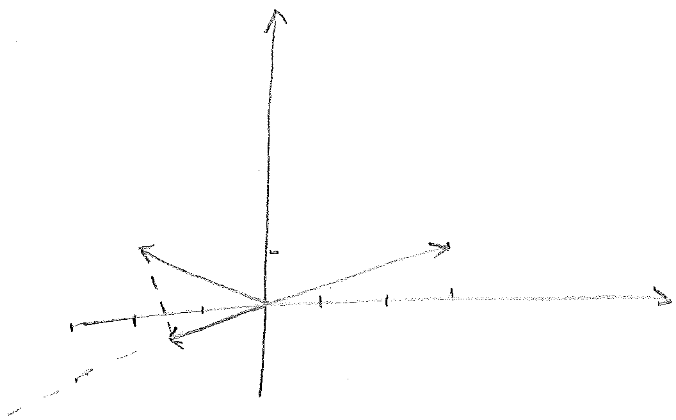
$$= \|v\| \cos\theta = \vec{v} \cdot \vec{u} = \vec{v} \cdot \frac{\vec{a}}{\|\vec{a}\|}$$

AND $\text{Proj}_{\vec{a}} \vec{v} = \text{Comp}_{\vec{a}} \vec{v} \cdot \vec{u} = \frac{(\vec{a} \cdot \vec{v})}{\|a\|^2} \cdot \vec{a}$

E.G.: $\vec{u} = \langle -2, 1 \rangle$, $\vec{v} = \langle 3, 1 \rangle$

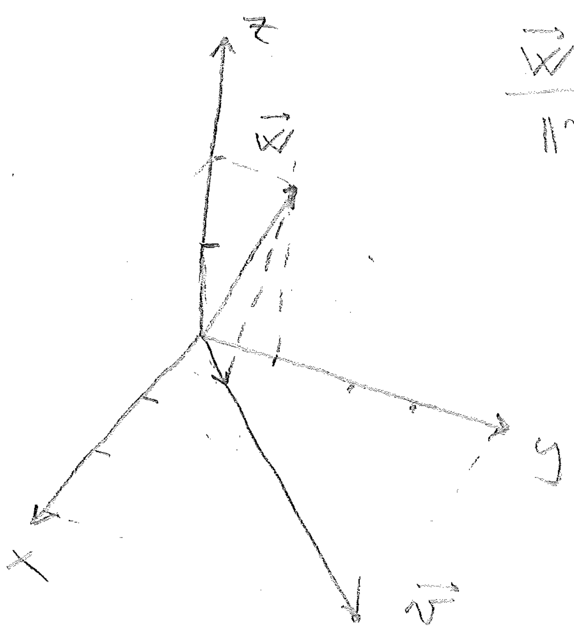
FIND $\text{PROJ}_{\vec{v}} \vec{u}$ AND SKETCH ALL THREE VECTORS

$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{-5}{10} \cdot \langle 3, 1 \rangle = \langle -\tfrac{3}{2}, -\tfrac{1}{2} \rangle$$



E.G.: $\vec{w} = \langle 0, 1, 2 \rangle$, $\vec{v} = \langle 3, 4, 0 \rangle$

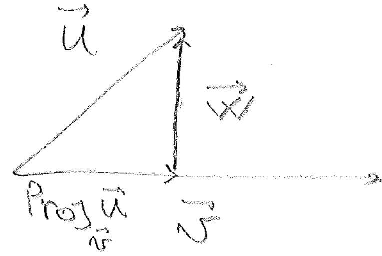
FIND $\text{PROJ}_{\vec{v}} \vec{w}$ AND SKETCH ALL THREE VECTORS



$$\frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \cdot \vec{v} = \frac{4}{25} \langle 3, 4, 0 \rangle =$$

$$= \langle \tfrac{12}{25}, \tfrac{16}{25}, 0 \rangle$$

ORTHOGONAL COMPONENT:



WE WANT TO WRITE

$\vec{u}$ AS $\text{Proj}_{\vec{v}}\vec{u} + \vec{w}$ WHERE

$\vec{w}$ IS ORTHOGONAL TO $\vec{v}$.

SILLY EQUALITY:

$$\left(\vec{u} - \text{proj}_{\vec{v}}\vec{u}\right) + \text{proj}_{\vec{v}}\vec{u} = \vec{u}$$

NOW, $\vec{v}\left(\vec{u} - \text{proj}_{\vec{v}}\vec{u}\right) =$

$$\vec{v}\cdot\vec{u} - \vec{v}\cdot\left(\frac{(\vec{v}\cdot\vec{u})}{\|\vec{v}\|^2}\,\vec{v}\right) = \vec{v}\cdot\vec{u} - \frac{\vec{v}\cdot\vec{v}}{\|v\|^2}(\vec{v}\cdot\vec{u})$$

$$= \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u} = 0 \quad !$$

SO THE PERPENDICULAR COMPONENT $\vec{w}$ IS EXACTLY $\vec{u} - \text{Proj}_{\vec{v}}\vec{u}$.

E.G. $\vec{w} = \langle 0, 1, 2\rangle$, $\vec{v} = \langle 3, 4, 0\rangle$

WE SAW THAT $\text{proj}_{\vec{v}}\vec{w} = \left\langle \frac{12}{25}, \frac{16}{25}, 0\right\rangle$

SO THE ORTHOGONAL COMPONENT IS

$$\vec{w} - \text{Proj}_{\vec{v}}\vec{w} = \langle 0, 1, 2\rangle - \left\langle \frac{12}{25}, \frac{16}{25}, 0\right\rangle = \left\langle \frac{-12}{25}, \frac{9}{25}, 2\right\rangle$$

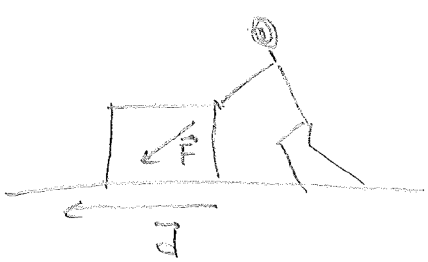
LET'S VERIFY:

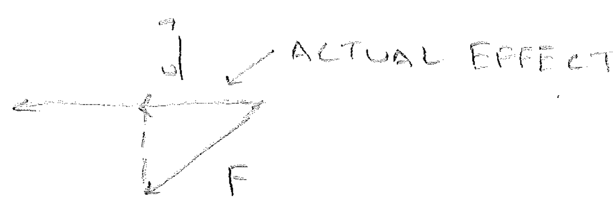
$$\left\langle -\frac{12}{25}, \frac{9}{25}, 2 \right\rangle \cdot \langle 3, 4, 0 \rangle =$$

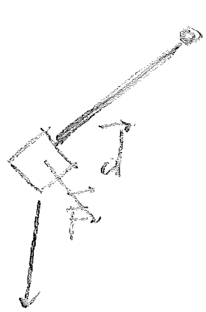
$$-\frac{36}{25} + \frac{36}{25} + 0 = 0 \quad \checkmark$$

FORCE AND WORK

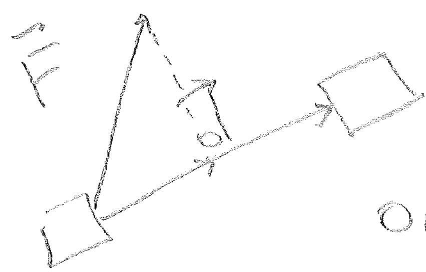
i) WHEN A BODY IS CONSTRAINED TO MOVE IN A GIVEN DIRECTION (FOR EXAMPLE, IT'S ON THE GROUND, ON A RAIL, ATTACHED TO A BAR) AND A FORCE IS APPLIED TO IT, ONLY THE COMPONENT OF THE FORCE ALONG THIS DIRECTION HAS AN EFFECT. THE CONSTRAINT COUNTERS THE REST:







ii) IF AN OBJECT UNDER A CONSTANT FORCE $\vec{F}$ IS DISPLACED BY A VECTOR $\vec{d}$ THEN THE WORK DONE BY $\vec{F}$ IS GOING TO BE



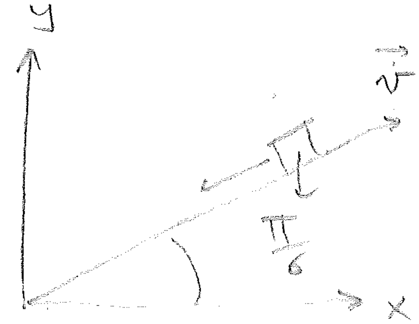
$$w = \vec{F} \cdot \vec{d}$$

ONLY THE PARALLEL PART COUNTS!

<u>E.G.:</u> AN OBJECT IS ON A 30° SLOPE. IF THE MASS IS 2 Kg:

a) WHAT IS THE OBJECT'S ACCELERATION DUE TO GRAVITY? (EXPRESSED AS A VECTOR $\vec{a}$)

b) WHAT IS THE WORK DONE BY GRAVITY AFTER THE OBJECT MOVED 5 METRES DOWN THE SLOPE?



a) $\vec{F} = m\vec{g} = \langle 0, -19.6 \rangle$

$\vec{v} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle$

$= \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$\text{Proj}_{\vec{v}} \vec{F} = -\frac{19.6}{2} \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$\vec{a} = \frac{-19.6}{4} \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

DIVIDE BY MASS