MORE EXERCISES ON INVERSE FUNCTIONS

- SHOW THAT THE FUNCTION $f(x) = e^{-x^2}$ IS INVERTIBLE ON $(0, +\infty)$
  - FIND $f^{-1}\left(\frac{1}{e^4}\right)$
  - FIND $(f^{-1})'\left(\frac{1}{e^4}\right)$

- SHOW THAT THE FUNCTION $f(x) = \dfrac{x^2+2}{x}$ IS INVERTIBLE ON $(-\infty, -2)$.
  - FIND $f^{-1}\left(-\frac{9}{2}\right)$
  - FIND $(f^{-1})'\left(-\frac{9}{2}\right)$

EXERCISES ON EXTREME VALUES

- $f(x) = x^{\frac{2}{3}} - \frac{2}{3}x$ ON $[-2, 2]$. FIND MINIMUM AND MAXIMUM

- $f(x) = \sin x \cos x$ ON $[-\pi, \pi]$. FIND MINIMUM AND MAXIMUM

EXTREME VALUES: A CRASH COURSE

① IF $f(x)$ IS CONTINUOUS ON A CLOSED INTERVAL $[a,b]$ THEN IT HAS A MAXIMUM AND MINIMUM VALUE ON $[a,b]$.

② $f(x)$ HAS A RELATIVE (OR LOCAL) MINIMUM OR MAXIMUM AT c IF, AFTER RESTRICTING $f(x)$ TO A SMALL NEIGHBOURHOOD $(c-h, c+h)$ OF c, $f(x)$ HAS A MAXIMUM/MINIMUM THERE.

③ WE WANT TO FIND THE MAXIMUM/MINIMUM OF $f(x)$ ON $[a,b]$. TO DO THIS WE MUST REDUCE TO CHECKING A (HOPEFULLY FINITE) LIST OF POINTS.

④ THE MAXIMUM/MINIMUM CAN BE AT

AN ENDPOINT

CHECK THE ENDPOINTS

A RELATIVE MAX/MIN

?

⑤ A CRITICAL NUMBER c IS A NUMBER WHERE $f'(c)=0$ OR $f'(c)$ DOES NOT EXIST. ALL RELATIVE MIN/MAX ARE AT CRITICAL NUMBERS

CHECK ALL CRITICAL NUMBERS

COMPARE THE VALUES OF $f(x)$ AT THESE NUMBERS TO FIND ABSOLUTE MIN/MAX

WARM-UP

① LET $f(x) = x\left(e^{-\frac{x}{3}+2}\right)$ ON $[\frac{1}{2}, 2]$,

FIND THE MINIMUM AND MAXIMUM OF $f(x)$.

SOL: $f(\frac{1}{2}) = \frac{e^{\frac{11}{6}}}{2} < f(2) = 2e^{\frac{4}{3}}$ $\quad f'(x) = e^{-\frac{x}{3}+2}(1-\frac{x}{3})$ ...

② LET $q(p) = e^{-\frac{p}{3}+2}$. USE THE ELASTICITY TO FIND THE OPTIMAL PRICE.

SOL: $\frac{dq}{dp} \cdot \frac{p}{q} = \frac{-e^{-\frac{p}{3}+2}}{3e^{-\frac{p}{3}+2}} \cdot p = -\frac{p}{3}$ $\quad p = 3$

③ LET $f(x) = (x-1)^{\frac{2}{3}} + 2$ ON $[0, 2]$. FIND THE ABSOLUTE MINIMUM AND MAXIMUM OF $f(x)$.

SOL: $f'(x) = \frac{2}{3}(x-1)^{\frac{2}{3}-1} = \frac{2}{3\sqrt[3]{x-1}}$ D.N.E. WHEN $x=1$

$f(0) = 3$, $f(2) = 3$, $f(1) = 2$

MIN $= 2$, MAX $= 3$

ONE SIDED DERIVATIVES:

WE'RE GIVEN A FUNCTION ON A CLOSED INTERVAL $[a, b]$. WHAT ARE $f'(a)$, $f'(b)$? DO THEY EXIST?

- THE FUNCTION f <u>DOES NOT EXIST</u> OUTSIDE OF THE GIVEN DOMAIN, NO MATTER IF THE FORMULA FOR f CAN BE USED OUTSIDE OF IT.
- WE CAN DO THE SAME THING WE DID FOR CONTINUITY, IN A SENSE

WE DEFINE

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \quad \text{IF IT EXISTS}$$

$$f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \quad \text{IF IT EXISTS}$$

THIS WAY WE CAN TELL WETHER a AND b ARE CRITICAL NUMBERS FOR f. IN MOST CASES IT'S THE SAME AS COMPUTING f' NORMALLY.

EXAMPLE:

i) $f(x) = \sqrt{1 - x^2}$ IN $[-1, 1]$ $\quad f'(-1) = \dfrac{-2x}{2\sqrt{1-x^2}}(-1) \text{"="} +\infty$

$f'(1) \text{"="} -\infty$ SO $-1, 1$ <u>ARE</u> CRIT VALUES

ii) $f(x) = x^2$ IN $[0, 1]$ $\quad f'(0) = 0$, $f'(1) = 2$

0 IS A CRIT VALUE, 1 IS NOT.

OK, NOW WE UNDERSTAND QUITE A BIT ABOUT CONTINUOUS FUNCTIONS ON CLOSED INTERVALS.

BUT

WHAT ABOUT OPEN INTERVALS? WHAT ABOUT ALL OF $\mathbb{R}$?

WE NEED:

INTERMEDIATE CURVE SKETCHING

TO BE ABLE TO CORRECTLY SKETCH A GENERAL FUNCTION, WE WILL NEED TO UNDERSTAND

- INFINITY-RELATED BEHAIVOIRS, I.E. ASYMPTOTES, LIMITS AT INFINITY.
- RELATIVE EXTREMES.
- INFORMATION CONVEYED BY THE SIGN OF FIRST AND SECOND DERIVATIVE.

1 ASYMPTOTES AND LIMITS

HOW DOES OUR FUNCTION BEHAVE FOR LARGER AND LARGER (POSITIVE OR NEGATIVE) VALUES OF x?

DEF: WE SAY THAT THE LIMIT OF $f(x)$ FOR $x \longmapsto +\infty$ (OR $-\infty$) IS L IF $f(x)$ GETS ARBITRARILY CLOSE TO L AS x GETS ARBITRARILY LARGE

- WE SAY THAT THE LIMIT OF $f(x)$ FOR $x \mapsto +\infty$ (OR $-\infty$) IS $\underline{+\infty}$ IF FOR ANY $M>0$ WE HAVE $f(x) > M$ FOR ALL $x>0$ BIG ENOUGH (RESP. FOR ALL $x<0$ BIG ENOUGH)
- WE SAY THAT THE LIMIT OF $f(x)$ FOR $x \mapsto +\infty$ (OR $-\infty$) IS $-\infty$ IF FOR ANY $M<0$ WE HAVE $f(x) < M$ FOR ALL $x>0$ BIG ENOUGH (RESP. FOR ALL $x<0$ BIG ENOUGH).

WE WRITE IT

$$\lim_{x \to +\infty} f(x) = \begin{cases} L \\ +\infty \\ -\infty \end{cases} \qquad \lim_{x \to -\infty} f(x) = \begin{cases} L \\ +\infty \\ -\infty \end{cases}$$

SO HOW DO WE COMPUTE THEM?

AS USUAL, LET'S START WITH THE EASY ONES.

$f(x)$ A POLYNOMIAL $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$

$a_0 \neq 0$,

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \underbrace{a_0 x^n}_{\text{GETS AS BIG AS WE WANT}} \underbrace{\left(1 + \frac{1}{x}\frac{a_1}{a_0} + \dots + \frac{1}{x^n}\frac{a_n}{a_n} \right)}_{\text{IS CLOSE TO 1 FOR BIG } x}$$

$$= \begin{cases} +\infty & a_0 > 0 \\ -\infty & a_0 < 0 \end{cases}$$

IF WE TAKE $\dfrac{1}{f(x)} = \dfrac{1}{a_0 x^n + \cdots + a_n}$

WE GET $\left.\lim_{x \to +\infty} \dfrac{1}{\underbrace{a_0 x^n + \cdots + a_n}_{\text{GETS ARBITRARILY BIG}}}\right\} = 0$

GETS ARBITRARILY SMALL

WHAT ABOUT $x \to -\infty$? WE NEED TO CHECK THE SIGN!

EXAMPLE:

$f(x) = -x^3 + 3x + 3$ $\quad \lim_{x \to -\infty} -x^3 + 3x + 3 = \lim_{x \to -\infty} -x^3 \overbrace{\left(1 - \frac{3}{x^2} - \frac{3}{x^3}\right)}^{\text{CLOSE TO 1}}$

$= \lim_{x \to -\infty} -x^3 = +\infty$!

NEGATIVE NUMBER

WE'LL BE MORE PRECISE IN A MOMENT

NEGATIVE COEFFICIENT

HOW ABOUT A QUOTIENT OF POLYNOMIALS?

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m} =$$

$$\lim_{x \to +\infty} \frac{a_0 x^n \left(1 + \frac{a_1}{a_0}\frac{1}{x} + \cdots + \frac{a_n}{a_0}\frac{1}{x^n}\right)}{b_0 x^m \left(1 + \frac{b_1}{b_0}\frac{1}{x} + \cdots + \frac{b_m}{b_0}\frac{1}{x^m}\right)} = \lim_{x \to +\infty} \frac{a_0}{b_0} x^{n-m}$$

$$= \begin{cases} \pm\infty & n > m \\ \dfrac{a_0}{b_0} & n = m \\ 0 & n < m \end{cases}$$

EXAMPLES:

- $$\lim_{x\to-\infty} \frac{x^3+3x+5}{-x^2+4x+3} = \lim_{x\to-\infty} \frac{x^3\left(1+\frac{3}{x^2}+\frac{5}{x^3}\right)}{-x^2\left(1-\frac{4}{x}-\frac{3}{x^2}\right)} = \lim_{x\to-\infty} -x = +\infty$$

- $$\lim_{x\to+\infty} \frac{4x^2+5x+7}{12x^2+3x+2} = \lim_{x\to+\infty} \frac{4x^2}{12x^2} \frac{\left(1+\frac{5}{4}\frac{1}{x}+\frac{7}{4x^2}\right)}{\left(1+\frac{3}{12x}+\frac{2}{12x^2}\right)} =$$

$$\lim_{x\to+\infty} \frac{4}{12} = \frac{4}{12}$$

- $$\lim_{x\to-\infty} \frac{\frac{x^2+3}{x}+\frac{x^3-4}{x^2}}{x^2+7} = \lim_{x\to-\infty} \frac{\frac{x^4+3x^2+x^4-4x}{x^3}}{x^2+7} =$$

$$\lim_{x\to-\infty} \frac{2x^4+3x^2-4x}{x^5+7x^3} = \lim_{x\to-\infty} \frac{2x^4}{x^5} \frac{\left(1+\frac{3}{2x^2}-\frac{4}{2x^3}\right)}{\left(1+\frac{7}{x^2}\right)} = \lim_{x\to-\infty} \frac{2}{x}$$

$$= 0$$

SO NOW WE KNOW LIMITS OF RATIONAL FUNCTIONS. WHAT CAN WE DO IN GENERAL?

WELL, SOME FUNCTIONS THAT WE KNOW HAVE A CLEAR LIMIT:

$$\lim_{x\to+\infty} \ln x = +\infty \qquad \lim_{x\to+\infty} e^x = +\infty, \quad \lim_{x\to-\infty} e^x = 0$$

$$\lim_{x\to+\infty} x^\alpha = \begin{cases} +\infty & \alpha>0 \\ 0 & \alpha<0 \end{cases}$$

BUT WE'D LIKE A MORE <u>GENERAL APPROACH</u>