DEFINITE AND INDEFINITE FORMS

WHEN DESCRIBING LIMITS WE CAN FIND AS RESULTS:

L IN $\mathbb{R}$, $+\infty$, $-\infty$.

WE WOULD LIKE TO INTRODUCE SOME "ALMOST ALGEBRAIC" OPERATIONS THAT WE CAN DO ON THEM

FOR THE SOLE PURPOSE OF COMPUTING LIMITS

WE SET:

- $L \pm \infty = \pm \infty$
- $0^{+\infty} = 0$
- IF $L > 0$, $L \cdot (+\infty) = +\infty$, $L \cdot (-\infty) = -\infty$
- IF $L < 0$, $L \cdot (+\infty) = -\infty$, $L \cdot (-\infty) = +\infty$
- $\frac{1}{\pm\infty} = 0$
- $\pm\infty \cdot \pm\infty = \pm\infty$,
- $+\infty^{L} = +\infty$
- $+\infty + \infty = +\infty$, $-\infty - \infty = -\infty$
- IF $L > 0$ $L^{\pm\infty} = \begin{cases} +\infty & \text{FOR } +\infty \\ 0 & \text{FOR } -\infty \end{cases}$

THE FORMULAS ABOVE, AND ALL THE ALREADY ADMISSIBLE FORMULAS WHERE NO $\pm\infty$ APPEARS, ARE CALLED DEFINITE FORMS

THE REMAINING ALGEBRAIC COMBINATIONS OF THESE SYMBLS, SUCH AS:

$$0 \cdot (\pm\infty), \quad \frac{0}{0}, \quad \underset{\uparrow *}{\frac{L}{0}}, \quad +\infty - \infty, \quad \frac{\pm\infty}{\pm\infty}, \quad \underset{\uparrow *}{\frac{\infty}{0}}, \quad 0^0, \quad +\infty^0, \quad 1^{\pm\infty}$$

ARE CALLED INDEFINITE FORMS. THE HAVE NO VALUE

NOTE: THE FORMS * ARE ONLY INDEFINITE BECAUSE OF THE SIGN.

SO WHAT'S THE USE FOR THESE?

THEOREM:

- ALL THE LIMIT RULES HOLD TRUE FOR LIMITS TO INFINITY <u>AS LONG</u> AS WE ONLY WORK WITH DEFINITE FORMS.

- MOREOVER, IF $\lim\limits_{x \to \pm\infty} g(x) = \mathcal{L} = \begin{cases} +\infty \\ -\infty \\ L \end{cases}$

THEN $\lim\limits_{x \to \pm\infty} f(g(x)) = \lim\limits_{x \to \mathcal{L}} f(x)$

EXAMPLE:

- $$\lim_{x \to +\infty} \frac{x^2+5}{x+2} = \lim_{x \to +\infty} \underbrace{\frac{x^2}{x}}_{\lim = \infty} \left(\frac{1+\frac{5}{x^2}}{1+\frac{2}{x}} \right) =$$

$$\lim_{x \to +\infty} \underbrace{x}_{\lim = \infty} \underbrace{\left(1+\frac{5}{x^2}\right)}_{\lim = 1} \underbrace{\left(\frac{1}{1+\frac{2}{x}}\right)}_{\substack{\lim = 1 \\ \text{by quotient rule}}} \underbrace{=}_{\substack{\infty \cdot 1 \cdot 1 \\ \text{definite} \\ \text{form}}} \lim_{x \to \infty} x \cdot 1 \cdot = +\infty$$

- $$\lim_{x \to +\infty} \underbrace{\ln(x^2+x) - \ln(x)}_{\substack{+\infty - \infty \\ \text{INDEFINITE} \\ \text{FORM}}} = \lim_{x \to +\infty} \ln\left(\frac{x^2+x}{x}\right) =$$

$$= \lim_{x \to +\infty} \ln(x+1) \underset{\underset{\lim\limits_{x \to +\infty} x+1 = +\infty}{\uparrow}}{=} \lim_{x \to +\infty} \ln(x) = +\infty$$

- $$\lim_{x \to +\infty} \underset{\underset{+\infty - \infty \quad \text{INDEF FORM}}{\uparrow}}{\sqrt{x^3 + x^2} - \sqrt{x^3}} = \lim_{x \to +\infty} \frac{\sqrt{x^3+x^2} - \sqrt{x^3}}{1} \frac{\left(\sqrt{x^3+x^2} + \sqrt{x^3}\right)}{\left(\sqrt{x^3+x^2} + \sqrt{x^3}\right)}$$

$$= \lim_{x \to +\infty} \frac{x^3 + x^2 - x^3}{\sqrt{x^3+x^2} + \sqrt{x^3}} = \lim_{x \to +\infty} \frac{x^2}{x^{\frac{3}{2}} \underset{\underset{\lim = 1}{\uparrow}}{\left(\sqrt{1+\frac{1}{x}} + 1\right)}} =$$

$$\lim_{x \to +\infty} \underset{\underset{\lim = \infty}{\uparrow}}{x^{\frac{1}{2}}} \cdot \underset{\underset{\lim = 1}{\uparrow}}{\frac{1}{\left(\sqrt{1+\frac{1}{x}} + 1\right)}} = +\infty$$

$\infty \cdot 1$

DEFINITE

- $$\lim_{x \to +\infty} \ln(x^2 + x) - \ln(x^2) = \lim_{x \to +\infty} \ln\left(\frac{x^2+x}{x^2}\right)$$

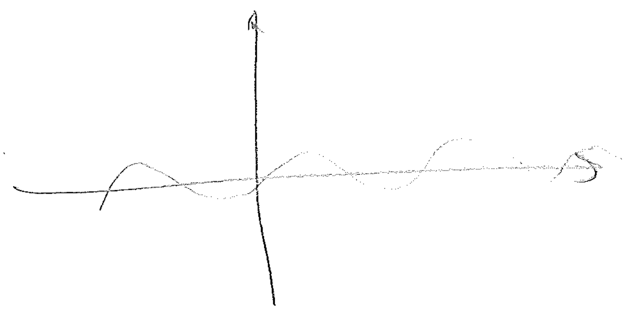
$$= \lim_{x \to +\infty} \underset{\underset{\lim = 1}{\uparrow}}{\ln\left(1 + \frac{1}{x}\right)} \underset{\underset{\text{COMP. RULE}}{\uparrow}}{=} \lim_{x \to 1} \ln(x) = 0$$

NOTE (IMPORTANT!):

A FUNCTION $f(x)$ MAY NOT HAVE A LIMIT AT $\pm\infty$!

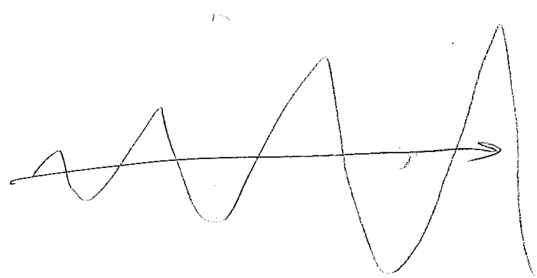
EXAMPLE:

$\lim_{x\to+\infty} \sin(x)$ DOES NOT EXIST



GOES BACK AND FORTH BETWEEN ± 1

$\lim_{x\to+\infty} x\sin(x)$ D.N.E.



EVEN WORSE OSCILLATING

$$\lim_{x\to+\infty} \frac{\sin(x)}{x} = 0$$

IT'S BOUNDED BY $\pm\frac{1}{x}$

NOTE (ALSO IMPORTANT!)

AN INDEFINIT FORM DOES NOT MEAN THAT THE lim DOES NOT EXIST. JUST THAT YOU MAY HAVE TO WORK HARDER TO FIND IT.

LIMITS GOING TO INFINITY AT A FINITE POINT

USING THE TERMINOLOGY WE DEVELOPED, WE CAN NOW ALSO CONSIDER THE SITUATION WHER A FUNCTION $f(x)$ "EXPLODES":

DEF:
WE SAY THE LIMIT OF $f(x)$ FOR $x \mapsto c$ IS $+\infty$ (RESP. $-\infty$) IF $f(x)$ GETS ARBITRARILY BIG AND POSITIVE (RESP. NEGATIVE) FOR x ARBITRARILY CLOSE TO c. WE WRITE

$$\lim_{x \to c} f(x) = +\infty \quad (\text{RESP. } -\infty)$$

DEF:
WE ANALOGOUSLY EXTEND THE NOTION OF ONE-SIDED LIMITS TO INCLUDE $\pm\infty$ AS POSSIBLE LIMITS. WE WRITE

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{OR} \quad \lim_{x \to c^-} = \pm\infty$$

REMARK: ONE-SIDED LIMITS ARE NECESSARY HERE; A SIMPLE EXAMPLE IS $f(x) = \frac{1}{x}$.

THM: ALL LIMIT RULES EXTEND TO THIS CASE, ASSUMING WE HAVE A DEFINITE FORM.

IT WILL BE USEFUL TO ADD TWO NEW SYMBOLS TO OUR TERMINOLOGY:

0^+ : TO INDICATE THAT A FUNCTION GOES TO 0 BUT <u>STAYS POSITIVE</u>

0^- : TO INDICATE THAT A FUNCTION GOES TO 0 BUT <u>STAYS NEGATIVE</u>

WE DEFINE 2 NEW <u>DEFINITE FORMS</u>:

$$\frac{1}{0^+} = +\infty \quad , \quad \frac{1}{0^-} = -\infty$$

MOREOVER, ALL DEFINITE FORMS WITH 0 ARE DEFINITE WHEN WE SUBSTITUTE 0^+ OR 0^-

EXAMPLE / EXERCISE:

- DOES $\lim_{x \to 0} \frac{1}{x}$ EXIST?
- WHAT ABOUT $\lim_{x \to 0^+} \frac{1}{x}$ AND $\lim_{x \to 0^-} \frac{1}{x}$?
- FIND $\lim_{x \to 1^-} \frac{x^2+2}{x^2-1}$
- FIND $\lim_{x \to 0^+} \ln(x)$

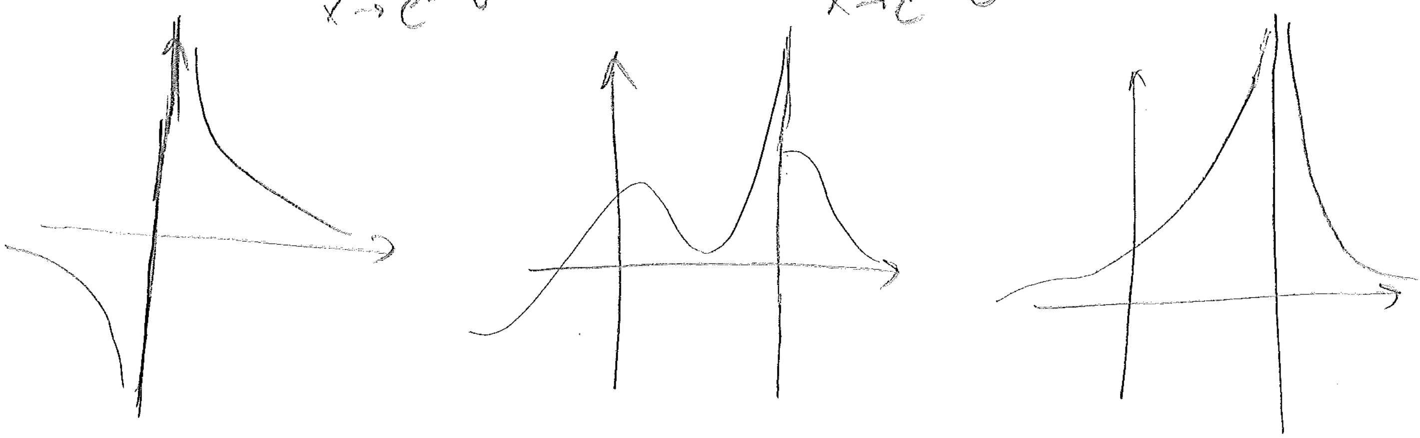
ASYMPTOTES:

SAYING $f(x)$ HAS AN ASYMPTOTE IS BASICALLY SAYING THAT THE "INFINITY" RELATED BEHAVOIR OF $f(x)$ CAN BE APPROXIMATED BY A LINE. WE WILL ONLY CONSIDER <u>HORIZONTAL</u> AND <u>VERTICAL</u> ASYMPTOTES.

DEF:

$f(x)$ HAS A <u>VERTICAL</u> ASYMPTOTE AT $x=c$ IF AT LEAST ONE OF THESE TWO STATEMENTS IS TRUE:

$$\lim_{x\to c^-} f(x) = \pm\infty \qquad \lim_{x\to c^+} f(x) = \pm\infty$$



$f(x)$ HAS A <u>HORIZONTAL</u> ASYMPTOTE AT $+\infty$ (RESP $-\infty$) IF

$$\lim_{x\to +\infty} f(x) = L \qquad \left(\text{RESP } \lim_{x\to -\infty} f(x) = L\right)$$

FOR A FINITE NUMBER L.