EXERCISES ON INVERSE FUNCTIONS

1. a) DEFINE WHAT AN INJECTIVE FUNCTION IS.

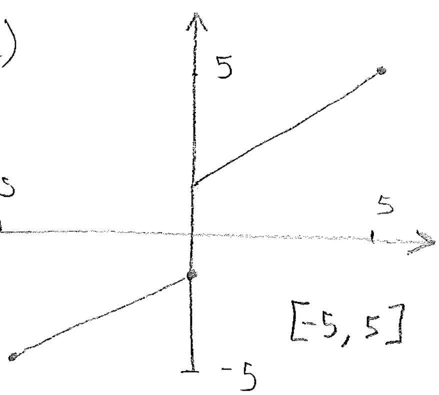
   b) DEFINE WHAT AN INVERSE FUNCTION IS, AND WHAT IT MEANS FOR A FUNCTION TO BE INVERTIBLE OR NOT.

   c) EXPLAIN WHY INJECTIVE FUNCTIONS ARE INVERTIBLE.

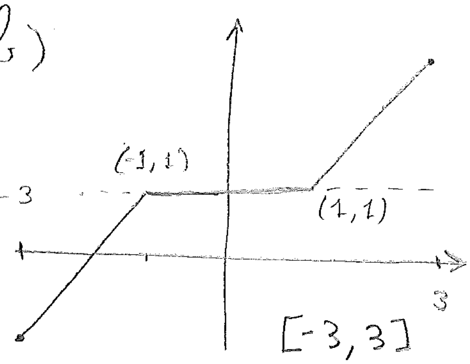
   d) GIVE A SUFFICIENT CRITERION FOR A FUNCTION TO BE INJECTIVE.

2. THREE OF THE FIVE FUNTIONS PICTURED BELOW ARE INVERTIBLE ON THE GIVEN INTERVAL. FIND THEM AND ASSIGN THEM THE CORRECT INVERSES IN THE NEXT SERIES OF PICTURES.

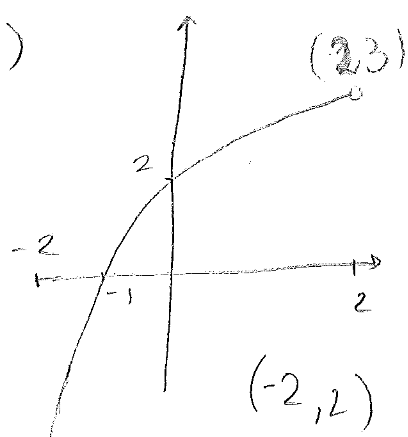
a)



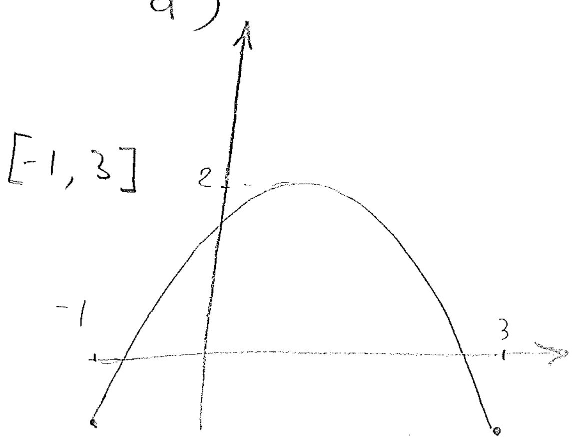
b)



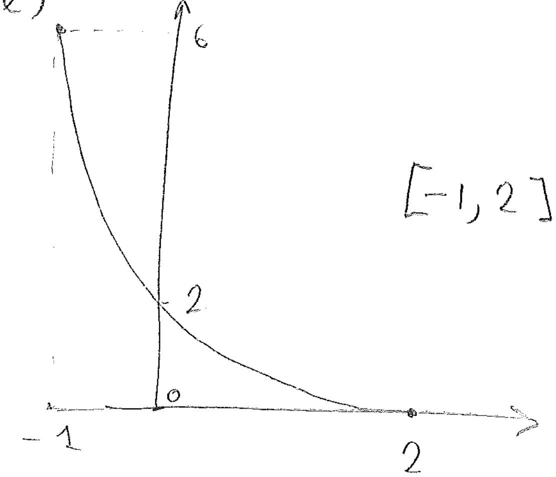
c)



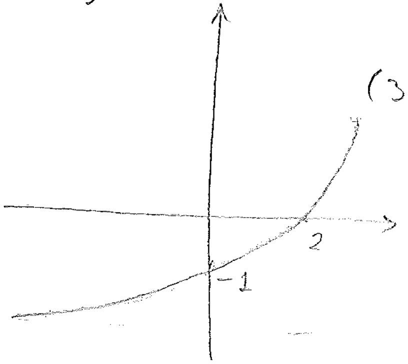
d)



e)

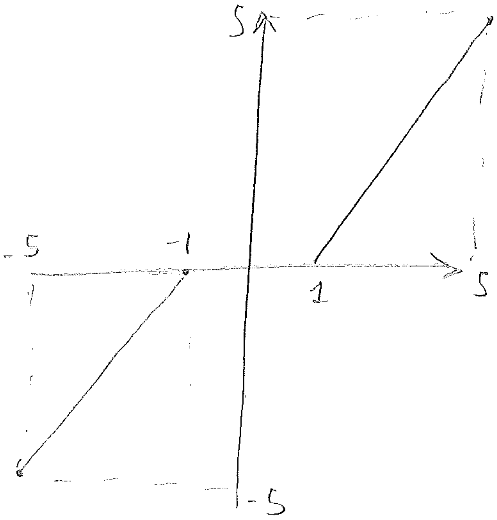


i)



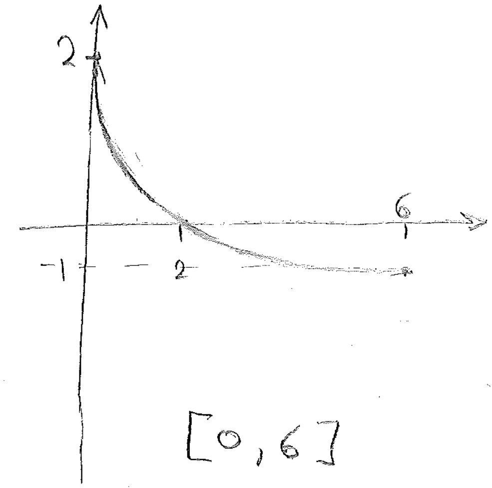
$(-\infty, 3)$

ii)



$[-5,-1] \cup (1,5]$

iii)



$[0,6]$

③ $f(x) = x^3 - 3x + 2$

a) FIND THE DERIVATIVE $f'(x)$

b) WRITE DOWN THE (MAXIMAL) INTERVALS WHERE $f(x)$ ADMITS AN INVERSE

c) RESTRICT $f(x)$ TO THE OPEN INTERVAL $(-1, 1)$, SO THAT IT HAS AN INVERSE $f^{-1}(x) = y(x)$. WHAT IS THE DOMAIN OF y?

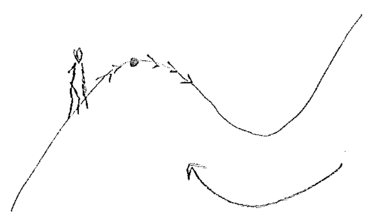
d) USING THE EXPLICIT FORMULA $y^3 - 3y + 2 = x$ FIND $y(2)$.

e) USING THE EXPLICIT FORMULA $y^3 - 3y + 2 = x$ AND IMPLICIT DIFF. FIND $y'(2)$.

d) VERIFY THAT YOU GET THE SAME RESULT USING $y' = \dfrac{1}{f'(y)}$.

LOOKING FOR MINIMA/MAXIMA

AS WE'VE SEEN, TAKING DECISIONS BY USING MATH GENERALLY INVOLVES TRYING TO MAXIMISE A FUNCTION (SUCH AS REVENUE) OR TO MINIMIZE A FUNCTION (SUCH AS COST). WHEN OUR FUNCTION ADMITS A DERIVATIVE, WE'LL GENERALLY WANT TO LOOK FOR A NUMBER WHERE IT CHANGES SIGN, FOLLOWING TH IDEA THAT

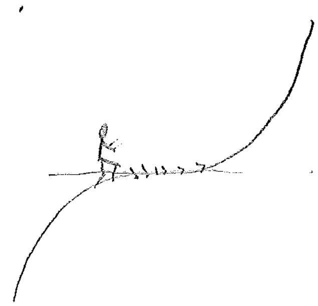


IF YOU WERE GOING UP AND THEN START GOING DOWN YOU JUST PASSED A PEAK (MAXIMUM)

IF YOU WERE GOING DOWN AN THEN START GOING UP YOU JUST CROSSED A VALLEY (MINIMUM)

SO WE'LL TRY TO LOOK FOR WHERE THE DERIVATIVE IS 0 (BY THE I.V.T. IF THE DERIVATIVE IS CONTINOUS IT MUST PASS THROUGH 0 WHEN IT CHANGES SIGN)

BEWARE:



THE DERIVATIVE BEING 0 DOES NOT MEAN YOU ARE AT A LOCAL MINIMUM OR MAXIMUM

(EX: $f(x) = x^3$)

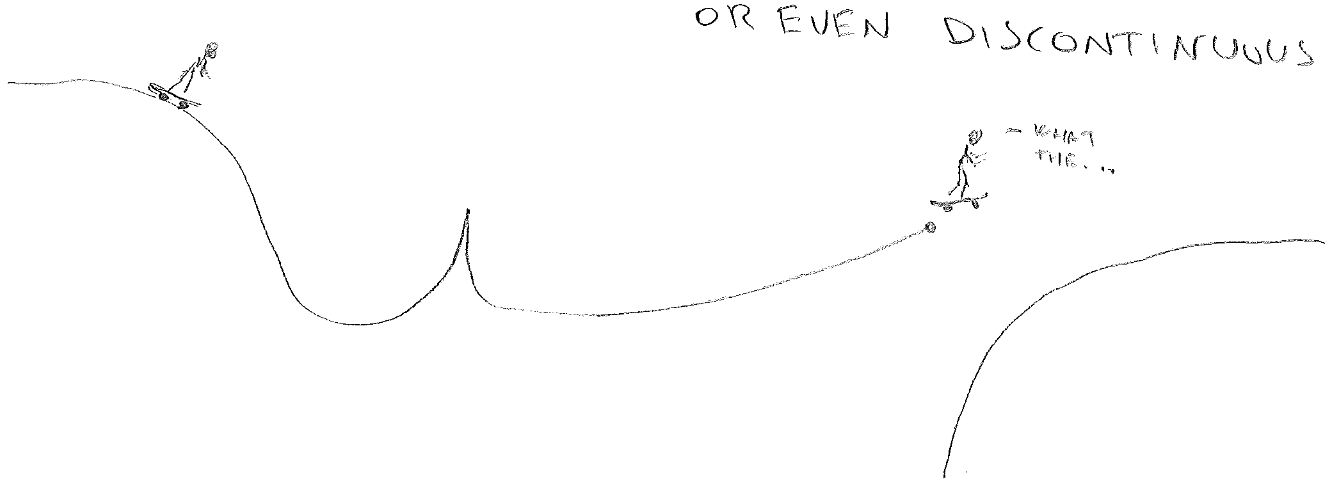
BUT:

THERE ARE A FEW MORE THINGS THAT CAN HAPPEN:

- OUR FUNCTION IS RESTRICTED TO A CLOSED INTERVAL, (FOR EXAMPLE, THERE ARE LEGAL LIMITATIONS TO PRICE)

- OUR FUNCTION IS NOT DIFFERENTIABLE AT SOME POINT

OR EVEN DISCONTINUOUS!



ALSO A FUNCTION MAY HAVE NO MINIMUM OR MAXIMUM AT ALL

$f(x) = \tan x$

WE NEED TO PUT THESE CONCEPTS IN ORDER.

DEF:

SUPPOSE $f(x)$ IS DEFINED ON AN INTERVAL I CONTAINING c.

- $f(c)$ IS THE MINUMUM (OR ABSOLUTE MINIMUM) OF f ON I IF $f(x) \geq f(c)$ FOR ALL x IN I
- $f(c)$ IS THE MAXIMUM (OR ABSOLUTE MAXIMUM) OF f ON I IF $f(x) \leq f(c)$ FOR ALL x IN I

THE MAXIMUM AND MINIMUM ARE CALLED THE EXTREME VALUES OF f ON I.

EXTREME VALUE THEOREM:

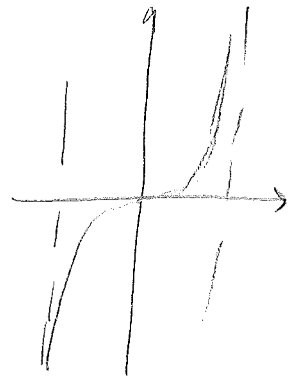
SUPPOSE $f(x)$ IS CONTINUOUS ON A CLOSED INTERVAL I. THEN f HAS BOTH A MINIMUM AND MAXIMUM ON I.

NOTE: BE CAREFUL WHEN USING THIS THOREM!

- THE INTERVAL **MUST** BE CLOSE

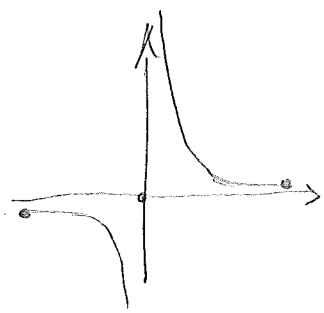
$f(x) = \tan x \quad I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

NO MAX, NO MIN



- $f(x)$ **MUST** BE CONTINUOUS ON THE FULL INTERVAL!

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$I = [-1, 1]$

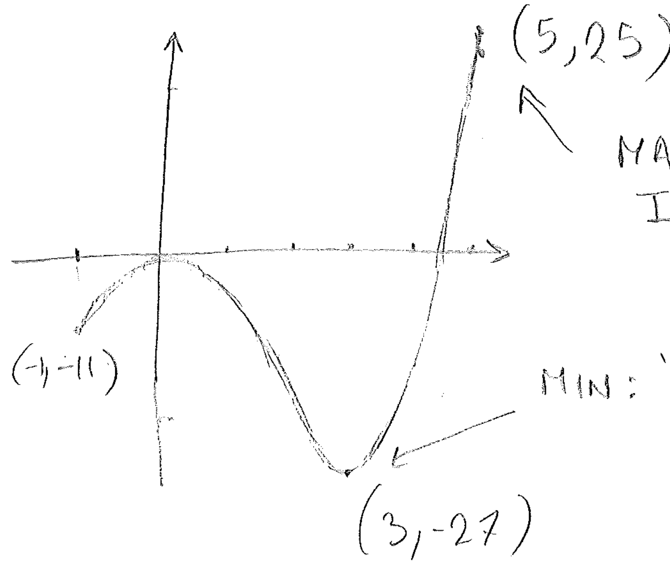
NO MAX, NO MIN.

BUT HOW DO WE **ACTUALLY** FIND THESE VALUES?

EXAMPLE:

$f(x) = 2x^3 - 9x^2$

$I = [-1, 5]$



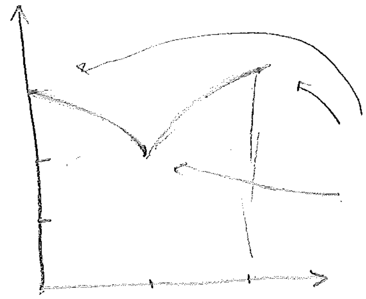
MAX: ENDPOINT OF I

MIN: "BOTTOM OF A VALLEY" ($f' = 0$)

EXAMPLE:

$f(x) = (x-1)^{2/3} + 2$

$I = [0,2]$



MAX: ENDPOINTS

MIN: CUSP

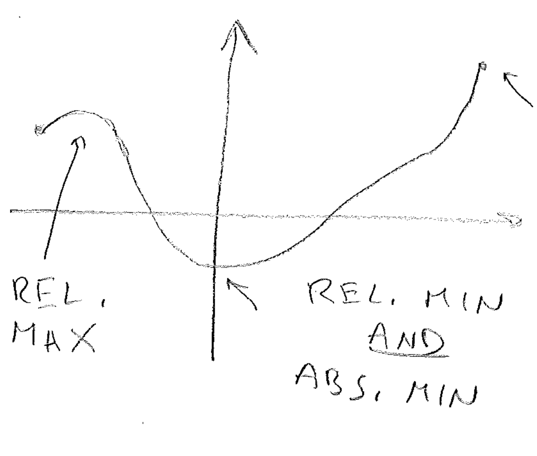
(f' DOES NOT EXIST)

DEF:

SUPPOSE f IS DEFINED ON AN INTERVAL I CONTAINING c.

- IF THERE IS AN OPEN INTERVAL $(c-h, c+h)$ CONTAINED IN I SUCH THAT $f(c)$ IS THE MINIMUM OF f ON $(c-h, c+h)$, THEN $f(c)$ IS A <u>RELATIVE</u> <u>MINIMUM</u>.
- IF THERE IS AN OPEN INTERVAL $(c-h, c+h)$ CONTAINED IN I SUCH THAT $f(c)$ IS THE MAXIMUM OF f ON $(c-h, c+h)$ THEN $f(c)$ IS A <u>RELATIVE</u> <u>MAXIMUM</u>.

NOTE THAT THE ENDPOINTS OF A CLOSED INTERVAL CAN NEVER GIVE US RELATIVE MINIMA/MAXIMA BUT CAN GIVE US <u>ABSOLUTE</u> ONES.



WE HAVE ALMOST ALL INGREDIENTS TO FIND MINIMA AND MAXIMA WITHOUT FAIL.

DEF:

SUPPOSE $f(x)$ IS DEFINED AT c. THE VALUE c IS A CRITICAL NUMBER (OR CRITICAL VALUE) IF

$f'(c) = 0$ OR $f'(c)$ DOES NOT EXIST.

THE POINT $(c, f(c))$ IS CALLED A CRITICAL POINT OF f.

THEOREM:

SUPPOSE f HAS A RELATIVE MINIMUM OR MAXIMUM AT c. THEN c IS A CRITICAL NUMBER OF f.

WE HAVE ALL WE NEED!

- HOW TO FIND MINIMA/MAXIMA ON A CLOSED INTERVAL.
  - SUPPOSE $f(x)$ IS CONTINUOUS ON I. BY THE E.V.T. WE KNOW THAT $f(x)$ HAS BOTH A MINIMUM AND A MAXIMUM ON I.
  - THESE CAN BE:
    a) ALSO A RELATIVE MIN/MAX OR
    b) AT AN ENDPOINT
  - SO WE JUST NEED TO EVALUATE $f(x)$ AT
    - ALL THE CRITICAL VALUES
    - THE ENDPOINTS
  - THE ABSOLUTE MAXIMUM OF f ON I IS THE MAXIMUM OF THESE VALUES, THE ABS. MIN. IS THE MIN. OF THESE VALUES.

EXAMPLE / EXERCISE:

FIND THE ABSOLUTE MAXIMUM AND ABSOLUTE MINIMUM OF

$$f(x) = \begin{cases} \sqrt{-25x} & x \leq 0 \\ \frac{x(-x+10)}{5} & x > 0 \end{cases}$$

ON $[-5, 8]$

SOL:

- ENDPOINTS: $-5, 10$
- CRITICAL VALUES:

$$f'(x) = \begin{cases} -\frac{5}{2\sqrt{-x}} & x < 0 \\ \text{NOT DEFINED} & x = 0 \\ \frac{-2x+10}{5} & x > 0 \end{cases}$$

SO THE CRIT VALUES ARE 0 AND 5

- $f(-5) = 5\sqrt{5}$, $f(8) = 16$, $f(0) = 0$, $f(5) = 5$

$f(-5)$: ABSOLUTE MAX

$f(0)$: RELATIVE MIN AND ABSOLUTE MIN

$f(5)$: RELATIVE MAX

- ABSOLUTE MAX $= 5\sqrt{5}$ AT -5, ABSOLUTE MIN $= 0$ AT 0.