WARM-UP EXERCISE

A) BACK TO THROWING STONES ON MARS

$h(T) = -1{,}85T^2 + 10T + 4 \qquad d(T) = 10T$

1. VERTICAL SPEED AS STONE HITS WATER?
2. DISTANCE THE STONE TRAVELED?
3. WE THROW A STONE ON MARS FROM THE WATER LEVEL, AT AN INITIAL VELOCITY OF
   - $v_\theta(0) = (\cos\theta \cdot 10, \sin\theta \cdot 10)$.

     WHAT IS THE MOTION $s(T) = (d(T), h(T))$?
   - HOW FAR WILL THE STONE GO?

(*) WHAT'S THE OPTIMAL θ?

(*) $h(T) = -1{,}85T^2 + 10\sin\theta\, T \qquad d(T) = 10\cos\theta\, T$

$$= T(-1{,}85T + 10\sin\theta)$$

$h(T_0) = 0 \sim T_0 = 0$ OR $-1{,}85T_0 = -10\sin\theta$

$T_0 = \dfrac{10}{1.85}\sin\theta$ SO $d(T_0) = \dfrac{100}{1.85}\sin\theta\cos\theta$

MAX DIST $\leftarrow \dfrac{d(d(T_0))}{d\theta} = 0 \sim 1 - 2(\sin\theta)^2 = 0$

$\theta = \arcsin\sqrt{\tfrac{1}{2}} = \dfrac{\pi}{4}$

RELATIVE RATES OF CHANGE

RATES OF CHANGE TAKE CARE OF MOST OF OUR NEEDS WHEN WE NEED TO SOLVE FOR EXAMPLE SIMPLE PHYSICS PROBLEMS, SUCH AS THE ONE WE JUST DID.

BUT

SOMETIMES WE NEED TO UNDERSTAND COMPARISONS WHERE THE R.O.C. BY THEMSELVES ARE NOT THAT SIGNIFICANT.

EXAMPLES:

1) A TOWN IS LOSING 500 CITIZENS/YEAR. HOW BAD IS IT?
2) YOUR FIRM'S PROFITS JUST INCREASED BY 3.000.000 \$/YEAR ARE YOU HAPPY?
3) A CERTAIN BRAND OF TRUCK HAS SEEN A PRICE INCREASE OF \$ 1000, WHILE A CERTAIN BRAND OF SOCKS HAS SEEN A PRICE INCREASE OF \$2. WHICH INCREASED MORE?

NONE OF THESE QUESTIONS MAKES SENSE. WE ARE GIVEN THE RATE OF CHANGE, BUT IT'S NOT ENOUGH TO GIVE AN ANSWER

1) IF OUR TOWN HAD 10.000 CITIZENS TO BEGIN WITH, WE'D BE LOOKING AT A $\frac{500}{10.000} = 5\%$ YEARLY LOSS IN POPULATION, THE TOWN COULD BE EMPTY IN 20 YRS.
IF OUR TOWN HAD 200.000 CITIZENS, WE'D BE LOOKING AT A $\frac{500}{200.000} = 0.25\%$ YEARLY LOSS IN POP., PRETTY ACCEPTABLE.

2) IF YOUR FIRM MADE \$ 10.000.000 LAST YEAR YOU HAVE A 30% INCREASE, GREAT NEWS. IF THEY MADE \$ 10.000.000.000, YOU'RE PROBABLY FIRED IF YOU'RE THE CEO

3) A TRUCK COSTS AT LEAST \$ 40.000, SO ITS PRICE INCREASED AT MOST 2.5 %, ON THE OTHER HAND, UNLESS THE SOCKS ARE IMPOSSIBLY EXPENSIVE, THEIR PRICE HAS GONE UP AT LEAST 10 %.

THE RELEVANT PARAMETER IN EACH OF THESE CASES (POPULATION, PROFIT, COST) IS THE <u>RELATIVE RATE OF CHANGE</u> (PER UNIT OF TIME)

$$\frac{f'(x)}{f(x)} = \frac{d \ln(f)}{dx} \longleftrightarrow \frac{\frac{\Delta f(x)}{\Delta x}}{f(x)} \quad \text{E.G.} \quad \frac{\frac{\text{COST}(2017)-\text{COST}(2016)}{1\text{ YR}}}{\text{COST (NOW)}}$$

NOTE THAT IF f HAS SOME UNITS ATTACHED

- $f(T)$ = POPULATION AT TIME T (IN PEOPLE)
- $f'(T)$ = POPULATION CHANGE $\frac{\text{PEOPLE}}{\text{YEARS}}$
- $\frac{f'(T)}{f(T)}$ = RELATIVE POP. CHANGE $= \frac{\text{PEOPLE}}{\text{YEARS}} \cdot \frac{1}{\text{PEOPLE}}$

$= \frac{1}{\text{YEARS}}$ (FRACTION OF POPULATION / YEARS)!

$\frac{f'(x)}{f(x)}$ IS JUST A RATE OF CHANGE, ONLY DEPENDING ON THE "INPUT" VARIABLE.

EXAMPLES:

- THE PRICE OF SOCKS FOLLOWS THE EQUATION

$$P(q) = -\frac{q^2}{10} + 2q + 4$$ WHERE WE PRODUCE

$q \cdot 1000$ SOCKS. WHAT'S THE RELATIVE RATE OF CHANGE AT $q = 5$?

$$\frac{P'(q)}{P(q)} = \frac{-\frac{q}{5} + 2}{-\frac{q^2}{10} + 2q + 4} \qquad \frac{P'(5)}{P(5)} = \frac{1}{-\frac{5}{2} + 10 + 4} = \frac{1}{\frac{23}{2}} = \frac{2}{23}$$

- A POPULATION GROWS AT THE RATE OF

$f(T) = 10^5 \cdot 1.1^{0.3T}$. WHAT'S THE RELATIVE RATE OF CHANGE?

$$\frac{f'(T)}{f(T)} = (\ln f(T))' = \left(5\ln 10 + 0.3T \ln(1.1)\right)' = (0.3)\ln(1.1)$$

MARGINAL COST

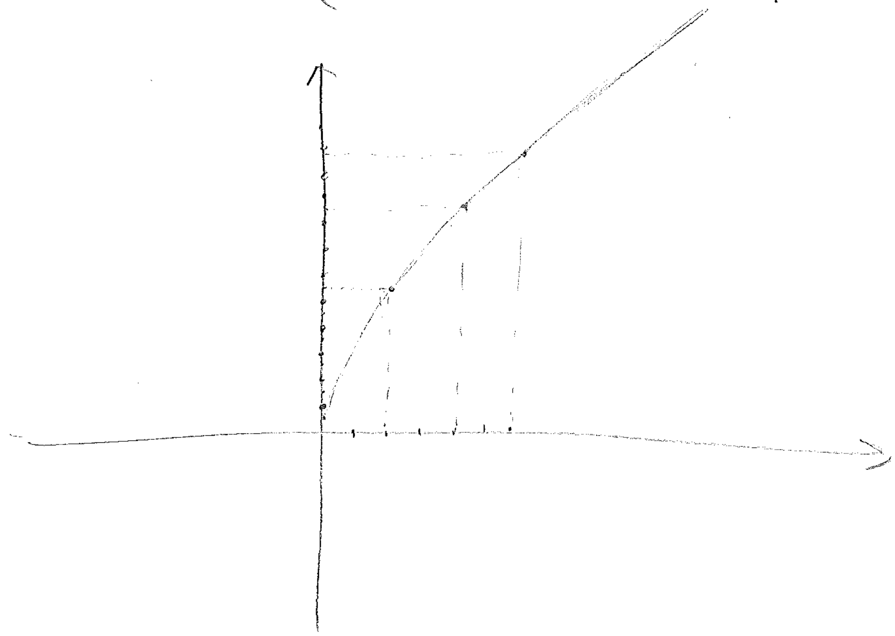
IN GENERAL THE FUNCTIONS WE FIND WHEN LOOKING AT REAL-LIFE BUSINESS PROBLEMS ARE NOT LINEAR. THIS FORCES US TO RECONSIDER SOME DEFINITIONS.

EXAMPLE:

PRODUCING ONE CAR FROM SCRATCH IS PRETTY EXPENSIVE. BUT AS THE NUMBER INCREASES WE GET CLOSER AND CLOSER TO INDUSTRIAL PRODUCTION WHICH HAS A MUCH LOWER COST PER UNIT.

A FUNCTION DESCRIBING THIS KIND OF COST WOULD BE FOR EXAMPLE: SAY TO PRODUCE $q \cdot 10$ CARS

$$C(q) = \left(-3^{\left(2-\frac{q}{2}\right)} + 2q + 10\right)1000$$



THE PRICE OF THE FIRST 10 CARS IS ≈ 68000 \$, BUT TO PRODUCE 1000 CARS WE SPEND ≈ 209.000 \$ SO THE PRICE IS DEFINITELY NOT LINEAR AT THE BEGINNING BUT IT STABILIZISES AT AROUND ≈ 2000 \$/CAR.

DEFINITION:

- THE <u>MARGINAL UNIT COST</u> (MUC) IS DEFINED AS $C(n+1) - C(n)$, WHERE n IS A POSITIVE INTEGER.
- THE <u>MARGINAL COST</u> (MC) IS DEFINED AS $\frac{dC}{dq}(q)$
- THE <u>AVERAGE UNIT COST</u> IS DEFINED AS

$$C_{avg} = \frac{C(q)}{q}.$$

THE MARGINAL UNIT COST IS THE ACTUAL COST OF PRODUCING A NEW UNIT, WHILE THE MARGINAL COST IS ITS CONTINUOUS COUNTERPART.

GENERAL PHILOSOPHY: CONTINUOUS IS BETTER THAN DISCRETE.

THE AVERAGE UNIT COST IS JUST THE AVERAGE COST OF ALL UNITS WE PRODUCED.

EXAMPLE:

$$C(q) = \left(-3^{\left(2-\frac{q}{2}\right)} + 2q + 10\right)\cdot 10000$$

$$MUC(m) = \left(\left(-3^{\left(2-\frac{m+1}{2}\right)} + 3^{\left(2-\frac{m}{2}\right)}\right) + 2\right)\cdot 10000$$

$$MC(q) = \left(\frac{\ln 3}{2}\, 3^{\left(2-\frac{q}{2}\right)} + 2\right)\cdot 10000$$

$$C_{avg}(q) = \left(\frac{-3^{\left(2-\frac{q}{2}\right)}}{q} + 2 + \frac{10}{q}\right)\cdot 10000$$

AT 1

$$MUC(1) = \left(\left(-3 + 3^{\frac{3}{2}}\right) + 2\right)\cdot 10000 \simeq (4.2)\cdot 10000$$

$$MC(1) = \left(\frac{\ln 3}{2}\cdot 3^{\frac{3}{2}} + 2\right)\cdot 10000 \simeq (4.85)\cdot 10.000$$

$$C_{avg}(1) = \left(-3^{\frac{3}{2}} + 2 + 10\right)\cdot 10000 \simeq (6.8)\cdot 10.000$$

FOR $m \gg 0$

$q \gg 0$ $\quad MUC(m) \simeq MC(q) \simeq C_{avg}(q) \simeq 2\cdot 10.000$

IF OUR COST FUNCTION IS REGULAR ENOUGH, THE MARGINAL COST IS A GOOD APPROXIMATION OF THE M.U.C.

CAN WE USE THE MC TO TAKE ACTUAL DECISIONS? YES!

SUPPOSE WE WANT TO MINIMIZE AVERAGE COST

- IF $MC(q) > C_{avg}(q)$ WE SHOULD ↓ PRODUCTION
- IF $MC(q) < C_{avg}(q)$ WE SHOULD ↑ PRODUCTION
- IF $MC(q) = C_{avg}(q)$ WE HAVE OPTIMAL PRODUCTION

WHY?

$$\frac{d\,C_{avg}}{dq} = \frac{d}{dq}\left(\frac{C(q)}{q}\right) = \frac{C'(q)\cdot q - C(q)}{q^2} = \frac{C'(q) - C_{avg}(q)}{q}$$

$$= \frac{MC(q) - C_{avg}(q)}{q}$$

SO IF $MC(q) > C_{avg}(q)$ C_{avg} IS INCREASING

" " $MC(q) < C_{avg}(q)$ C_{avg} IS DECREASING

WRAP-UP EXERCISE:

(1) DUE TO A WEIRD TAX SYSTEM, THE COST OF PRODUCING TEDDY BEARS IN ITALY IS AS FOLLOWS:

$$C(q) = q^3 - 3q^2 + q + 5$$

WHERE WE PAY $C(q) \cdot 1000$ €, PRODUCE $q \cdot 1000$ UNITS.

TRY TO FIND THE PRODUCTION VALUE THAT MINIMIZES THE AVERAGE COST.

(FIND AN INTERVAL CONTAINING OPTIMAL VALUE)

SOL:

$$C_{avg}(q) = q^2 - 3q + 1 + \frac{5}{q} \qquad MC(q) = 3q^2 - 6q + 1$$

$$C_{avg}(q) = MC(q) \sim q^2 - 3q + 1 + \frac{5}{q} = 3q^2 - 6q + 1$$

$$q^3 - 3q^2 + \cancel{q} + 5 = 3q^3 - 6q^2 + \cancel{q} \sim 2q^3 - 3q^2 - 5 = 0$$

WE USE I.V.T TO FIND AN INTERVAL:

$q = 2$ $\quad 16 - 12 - 5 = -1 < 0$

$q = 3$ $\quad 54 - 27 - 5 = 22 > 0$

OPTIMAL q BETW. 2 AND 3