WARM-UP EXERCISE

- $h(x) = e^{f(x)} + (f(x))^2$

  KNOWING $f(1) = 2$, $f'(1) = 5$ find $h'(1)$

- FIND THE EQUATION OF THE TANGENT TO THE CURVE $x^3 + xy^2 + y^3 = 13$ AT $(1,2)$. SIMPLIFY.

SOL

- $h'(x) = f'(x)e^{f(x)} + 2f'(x)f(x)$

  $h'(1) = 5e^2 + 20$

- $\frac{d}{dx}(x^3 + xy^2 + y^3) = \frac{d}{dx} 13 = 0$

  $3x^2 + y^2 + 2xy'y + 3y'y^2 = 0$

  $y'(2xy + 3y^2) = -3x^2 + y^2$

  $y' = \frac{-3x^2 + y^2}{2xy + 3y^2}$ At $(1,2)$ $y' = \frac{-3+4}{2+12}$

  $= \frac{1}{14}$

EXAMPLE: LOGARITHMIC DIFFERENTIATION

SOMETIMES THE "IMPLICIT APPROACH" CAN HELP US ALSO IN THE EXPLICIT CASE.

SAY WE HAVE TO FIND THE DERIVATIVE OF $y = x^x$. THEN WE CAN EXTRACT THE LOGARITHM

$\ln(y) = \ln(x^x) = x\ln(x)$, DIFFERENTIATE,

$$\frac{d\ln(y)}{dx} = \frac{d}{dx}x\ln(x) \sim \frac{y'}{y} = 1 + \ln(x)$$

SOLVE FOR y'

$y' = y(1+\ln(x))$ AND SUBSTITUTE $y = x^x$

$y' = x^x(1+\ln(x))$

IN GENERAL

$y = u(x)^{v(x)} \sim \ln y = v(x)\ln(u(x))$ DIFFERENTIATE

$$\frac{y'}{y} = \frac{v(x)}{u(x)} + v'(x)\ln(u(x)) \sim y' = y\left(\frac{v(x)u'(x)}{u(x)} + v'(x)\ln(u(x))\right)$$

SUBSTITUTE $y' = u(x)^{v(x)}\left(\frac{v(x)u'(x)}{u(x)} + v'(x)\ln(u(x))\right)$

ANOTHER EXAMPLE

$$y = \frac{u(x)^n v(x)^m}{h(x)^k} \sim \ln(y) = n\ln(u(x)) + m\ln(v(x)) - k\ln(h(x))$$

$$\frac{y'}{y} = n\frac{u'(x)}{u(x)} + m\frac{v'(x)}{v(x)} - k\frac{h'(x)}{h(x)} \quad y' = y\left(n\frac{u'(x)}{u(x)} + m\frac{v'(x)}{v(x)} - k\frac{h'(x)}{h(x)}\right)$$

SO $y' = \dfrac{u(x)^n v(x)^m}{h(x)^m}\left(n\dfrac{u'(x)}{u(x)} + m\dfrac{v'(x)}{v(x)} - k\dfrac{h'(x)}{h(x)}\right)$

EXAMPLES:

- $f(x) = \sqrt{x}^{(\cos x)^2}$ $\quad f'(x) = \sqrt{x}^{(\cos x)^2}\left(\dfrac{\cos(x)^2 \cdot \frac{1}{2}\frac{1}{\sqrt{x}}}{\sqrt{x}} - 2\cos x \sin x \underbrace{\ln(\sqrt{x})}_{= \frac{1}{2}\ln(x)}\right)$

- $f(x) = \dfrac{\sin(x)^3 \ln(x)^3}{\cos(x)^2}$ $\quad f'(x) = \dfrac{\sin(x)^3 \ln(x)^3}{\cos(x)^2}\left(3\dfrac{\cos x}{\sin x} + 3\dfrac{1}{x\ln x} + 2\dfrac{\sin x}{\cos x}\right)$

HIGHER ORDER IMPLICIT DERIVATIVES

WITH SOME EFFORT WE CAN COMPUTE HIGHER IMPLICIT DERIVATIVES.

EXAMPLE:

- GIVEN $x^2 + y^2 = 1$ FIND A FORMULA FOR y''

WE KNOW SINCE EARLIER THAT

$y' = \dfrac{-x}{y}$. THEN

SUBSTITUTE y'

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{-y + y'x}{y^2} \overset{\downarrow}{=} \frac{-y - \frac{x^2}{y}}{y^2}$$

ONE MORE EXAMPLE: $y^{\frac{2}{3}} + x^{\frac{2}{3}} = 8$. FIND y'

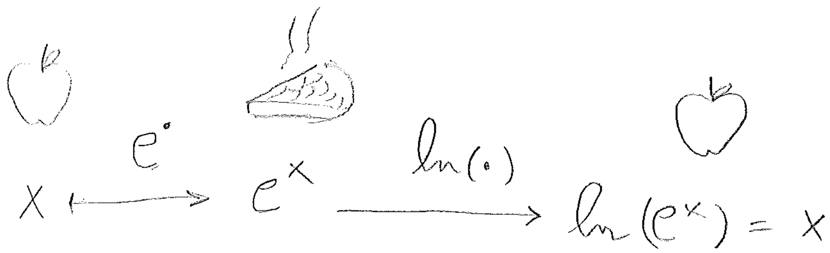
AT $(8,8)$. $\quad y'\frac{2}{3}y^{\frac{2}{3}-1} + \frac{2}{3}x^{\frac{2}{3}-1} = 0 \rightsquigarrow y' = -\dfrac{x^{\frac{2}{3}-1}}{y^{\frac{2}{3}-1}}$

SO AT $(8,8)$ $\quad y' = -\dfrac{8^{-\frac{1}{3}}}{8^{-\frac{1}{3}}} = -1.$

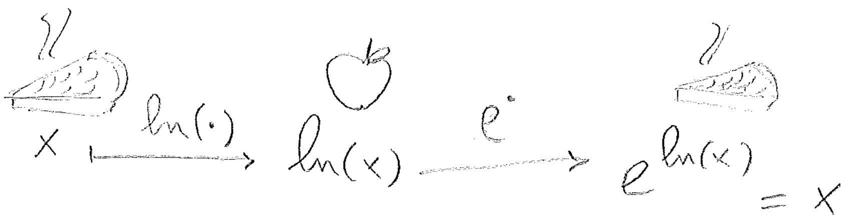
INVERSE FUNCTIONS (AND THEIR DERIVATIVES)

WE ALL KNOW THE RELATION $e^{\ln x} = \ln e^x = x$

IT TELLS US THAT THE EXPONENTIAL FUNCTION AND NATURAL LOGHARITHM FUNCTION ARE INVERSE TO EACH OTHER



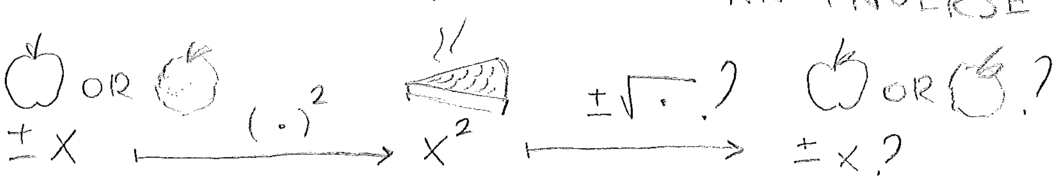
$$x \xmapsto{e^{\circ}} e^x \xrightarrow{\ln(\cdot)} \ln(e^x) = x$$



$$x \xmapsto{\ln(\cdot)} \ln(x) \xrightarrow{e^{\circ}} e^{\ln(x)} = x$$

SO IF $g(x)$ IS THE INVERSE TO $f(x)$, THIS MEANS THAT GIVEN A: $y = f(x)$, $g(y)$ TELLS US EXACTLY WHICH x IT CAME FROM.

CAN WE ALWAYS FIND AN INVERSE FUNCTION? NO!



$$\pm x \xmapsto{(\cdot)^2} x^2 \xrightarrow{\pm\sqrt{\cdot}\,?} \pm x\,?$$

WE NEED THE INVERSE FUNCTION TO BE, WELL, A FUNCTION, SO THERE MUST BE ONLY ONE VALUE OF x SUCH THAT $f(x) = y$

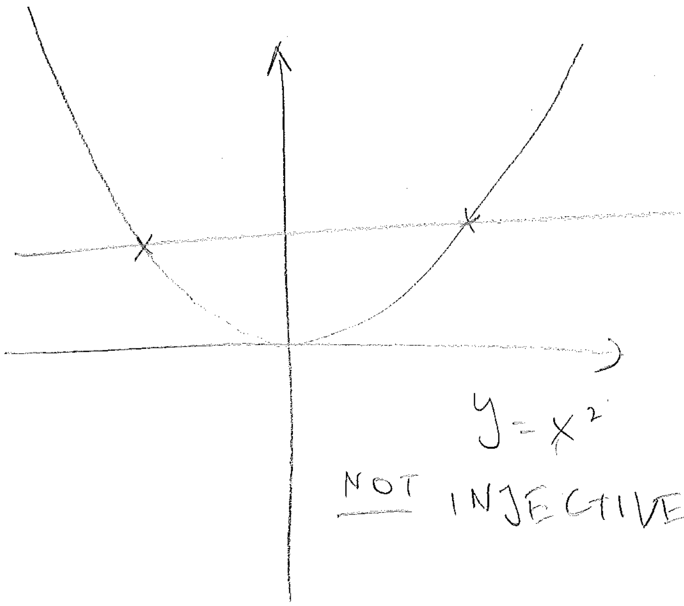
DEFINITION

A FUNCTION IS CALLED INJECTIVE IF FOR ANY y IN THE CODOMAIN (OR RANGE) OF f THERE IS AT MOST ONE x WITH $f(x) = y$

HOW DO WE TELL IF A FUNCTION IS INJECTIVE?

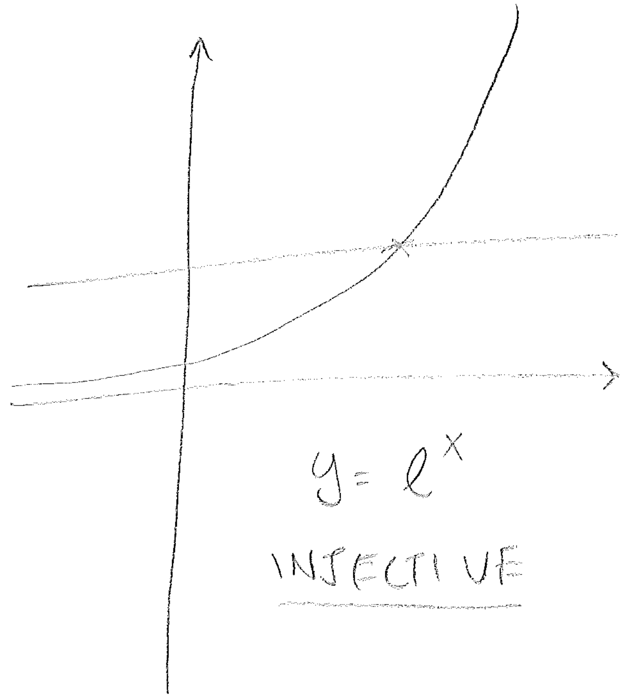
GRAPHICALLY, INJECTIVE MEANS

"ANY HORIZONTAL LINE INTERSECTS $f(x)$ IN <u>AT MOST</u> ONE POINT"



$y = x^2$

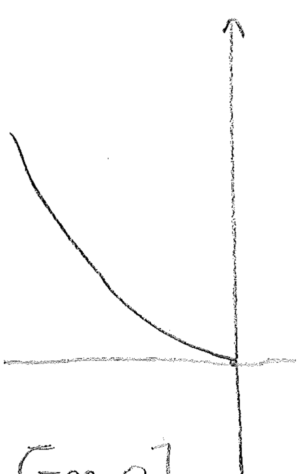
<u>NOT</u> INJECTIVE



$y = e^x$

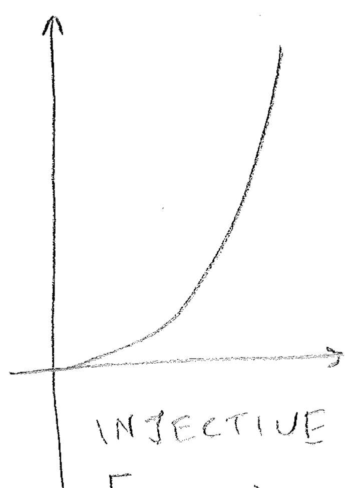
<u>INJECTIVE</u>

BEING INJECTIVE DEPENDS <u>ON THE DOMAIN</u>; FOR EXAMPLE $f(x) = x^2$ IS INJECTIVE ON $x \leq 0$ <u>OR</u> $x \geq 0$ BUT NOT BOTH



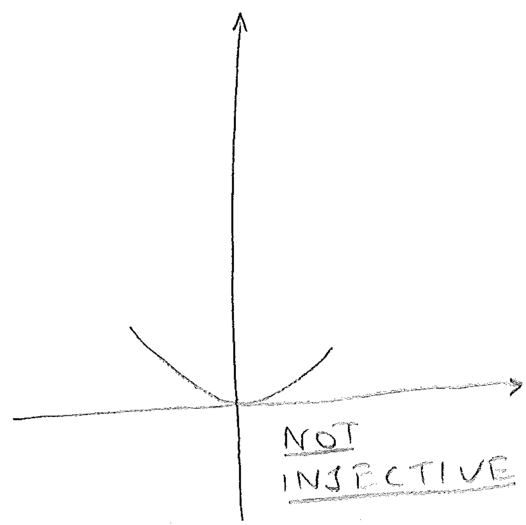
$(-\infty, 0]$

INJECTIVE



INJECTIVE

$[0, +\infty)$



<u>NOT INJECTIVE</u>

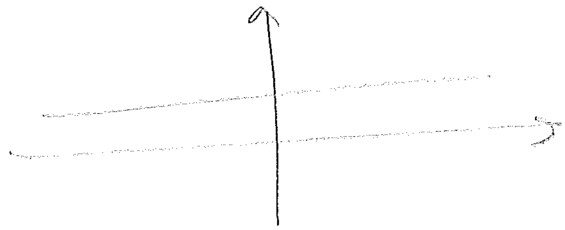
$[-a, a]$

ON $(-\infty, 0]$ THE INVERSE TO $x \mapsto x^2$ IS $x \longmapsto -\sqrt{x}$

ON $[0, +\infty)$ THE INVERSE TO $x \mapsto x^2$ IS $x \longmapsto \sqrt{x}$

CAN WE ALWAYS RESTRICT THE DOMAIN TO MAKE A FUNCTION INJECTIVE? NO!

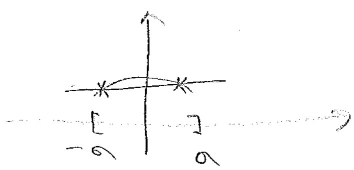
$f(x) = 1$



NEVER INJECTIVE

WE WOULD OFTEN LIKE TO HAVE A FUNCTION THAT IS INJECTIVE <u>AROUND</u> A POINT, BUT THAT'S NOT ALWAYS POSSIBLE EITHER

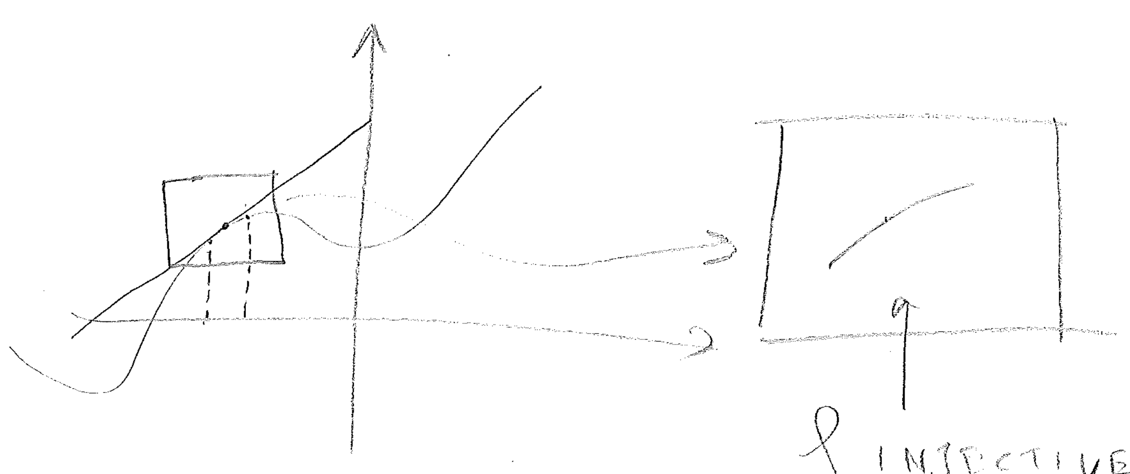
$f(x) = \cos x$



WILL NEVER BE INJECTIVE ON $[-a, a]$

<u>BUT</u> THERE IS A CONDITION THAT ALLOWS US TO SAY THAT A FUNCTION IS (LOCALLY) INJECTIVE

<u>THEOREM</u>: IF $f'(x)$ IS CONTINUOUS ON (a, b), AND $f'(x_0) > 0$ (OR $f'(x_0) < 0$) ON ALL (a, b), THEN $f(x)$ IS <u>INJECTIVE</u> ON ALL (a, b)



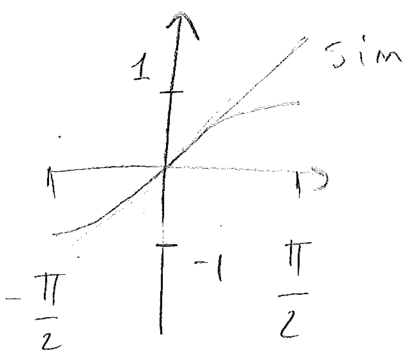
f INJECTIVE

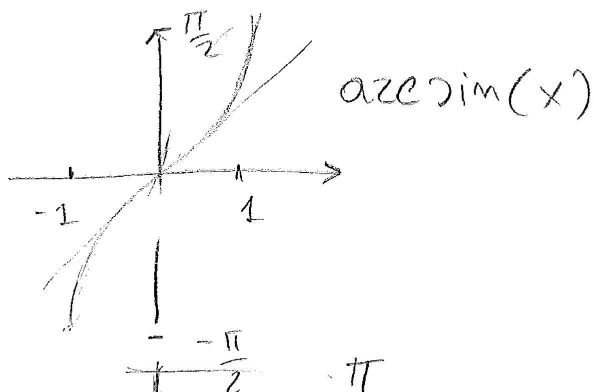
IF $f(x)$ IS <u>INJECTIVE</u> ON (a, b) THEN IT HAS AN <u>INVERSE</u>, WHICH WE CALL $f^{-1}(x)$.

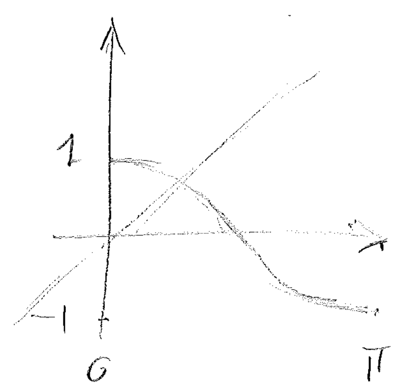
INVERSES OF TRIG FUNCTIONS

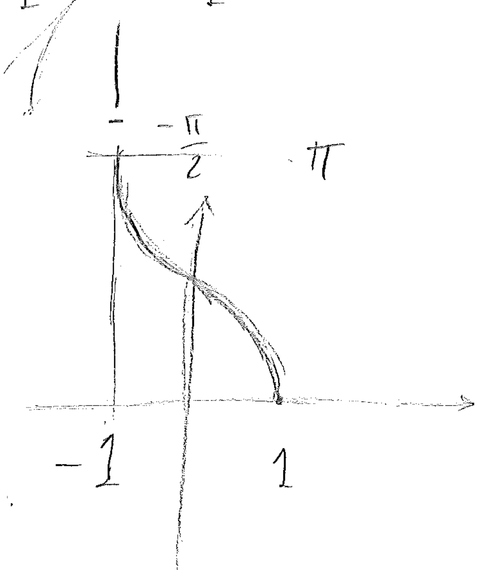
CONSIDER THE FUNCTIONS
$\sin(x)$ AND $\cos(x)$. WE HAVE
$(\sin(x))' = \cos(x)$ AND $(\cos(x))' = -\sin(x)$
THE THEOREM TELLS US THAT ON
$[-\frac{\pi}{2}, \frac{\pi}{2}]$ $\sin(x)$ IS <u>INJECTIVE</u> AND THUS <u>INVERTIBLE</u>
AND ON
$[0, \pi]$ $\cos(x)$ IS <u>INJECTIVE</u> AND THUS <u>INVERTIBLE</u>.

WE CALL
$\arcsin(x) = \sin^{-1}(x)$ AND $\arccos(x) = \cos^{-1}(x)$
THEIR INVERSE FUNCTIONS ON THESE DOMAINS









WE'RE REFLECTING ON THE LINE $y = x$!

HOW DO WE FIND THEIR DERIVATIVES?

WITH IMPLICIT DIFFERENTIATION!

$y = \sin^{-1}(x)$ $\quad \sin(y) = x \quad (\sin(y))' = (x)'$

$y' \cos y = 1 \quad y' = \frac{1}{\cos y} \quad y' = \frac{1}{\sqrt{1 - \sin y^2}} = \frac{1}{\sqrt{1 - x^2}}$

$y = \cos^{-1}(x) \quad \cos(y) = x \quad (\cos(y))' = (x)'$

$-y' \sin(y) = 1 \quad y = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - (\cos y)^2}} = \frac{-1}{\sqrt{1 - x^2}}$

CAN WE DO THIS IN GENERAL?

$y = f^{-1}(x) \quad f(y) = x \quad y' f'(y) = 1$

$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$

WRAP-UP EXERCISES

- f, g DIFFERENTIABLE, $h(x) = f(g(x))$
  TANGENT TO g AT $(4,7)$ IS $y = 3x - 5$
  " " f AT $(7,9)$ IS $y = -2x + 23$
  FIND THE TANGENT TO h' AT $x = 4$
  (EASIER: FIND $h'(4)$)

- FIND $\frac{dy}{dx}$, WHERE $y = x^{\ln x}$

- FIND $f'(0)$, WHERE $f(x) = \tan^{-1}(5x)$
  $(= \arctan(5x))$

- CONSIDER THE CURVE $x^2 + y^3 - 2xy = 0$ ASSUME $(1,1)$ LIES ON THE CURVE, AND NEARBY SOLUTIONS SATISFY $y = f(x)$ FOR SOME $f(x)$.
  FIND $f(1)$ AND $f'(1)$

- FIND a TO MAKE

$$f(x) = \begin{cases} x^2 + a & x \le e \\ 3a \ln x & x > e \end{cases}$$

  CONTINUOUS FOR ALL x

- USE THE I.V.T. TO PROVE THAT
  $e^x - x^2$ HAS A SOLUTION