WRAP-UP EXERCISES

- f, g DIFFERENTIABLE, $h(x) = f(g(x))$
  TANGENT TO g AT $(4,7)$ IS $y = 3x - 5$
  " " f AT $(7,9)$ IS $y = -2x + 23$
  FIND THE TANGENT TO h' AT $x = 4$
  (EASIER: FIND $h'(4)$)

- FIND $\frac{dy}{dx}$, WHERE $y = x^{\ln x}$

- FIND $f'(0)$, WHERE $f(x) = \mathrm{Tan}^{-1}(5x)$
  $(= \arctan(5x))$

- CONSIDER THE CURVE $x^2 + y^3 - 2xy = 0$ ASSUME $(1,1)$ LIES ON THE CURVE, AND NEARBY SOLUTIONS SATISFY $y = f(x)$ FOR SOME $f(x)$.
  FIND $f(1)$ AND $f'(1)$

- FIND a TO MAKE

$$f(x) = \begin{cases} x^2 + a & x \le e \\ 3a \ln x & x > e \end{cases}$$

  CONTINUOUS FOR ALL x

- USE THE I.V.T. TO PROVE THAT
  $e^x - x^2$ HAS A SOLUTION

INVERSES OF TRIG FUNCTIONS

CONSIDER THE FUNCTIONS

$\sin(x)$ AND $\cos(x)$. WE HAVE

$(\sin(x))' = \cos(x)$ AND $(\cos(x))' = -\sin(x)$

THE THEOREM TELLS US THAT ON

$[-\frac{\pi}{2}, \frac{\pi}{2}]$ $\sin(x)$ IS INJECTIVE AND THUS INVERTIBLE

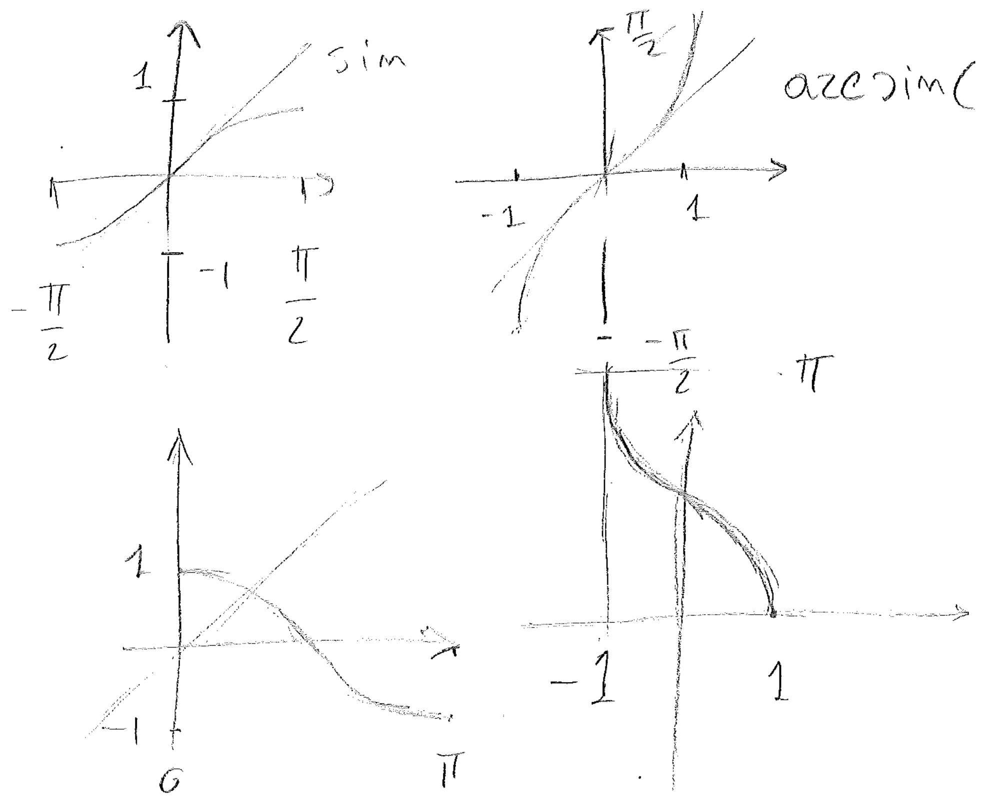
AND ON

$[0, \pi]$ $\cos(x)$ IS INJECTIVE AND THUS INVERTIBLE.

WE CALL

$\arcsin(x) = \sin^{-1}(x)$ AND $\arccos(x) = \cos^{-1}(x)$

THEIR INVERSE FUNCTIONS ON THESE DOMAINS



WE'RE REFLECTING ON THE LINE $y = x$!

HOW DO WE FIND THEIR DERIVATIVES?

WITH IMPLICIT DIFFERENTIATION!

$y = \sin^{-1}(x) \qquad \sin(y) = x \qquad (\sin(y))' = (x)'$

$y'\cos y = 1 \qquad y' = \frac{1}{\cos y} \qquad y' = \frac{1}{\sqrt{1-\sin y^2}} = \frac{1}{\sqrt{1-x^2}}$

$y = \cos^{-1}(x) \qquad \cos(y) = x \qquad (\cos(y))' = (x)'$

$-y'\sin(y) = 1 \qquad y = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-(\cos y)^2}} = -\frac{1}{\sqrt{1-x^2}}$

CAN WE DO THIS IN GENERAL?

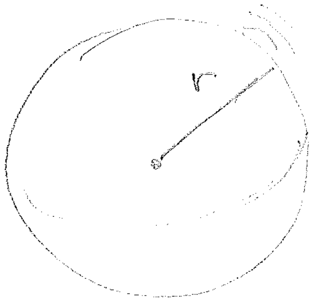
$y = f^{-1}(x) \qquad f(y) = x \qquad y'f'(y) = 1$

$y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$

RELATED RATES PROBLEMS

1) AIR IS BEING PUMPED INTO A SPERICAL BALLOON AT A RATE OF $100 \frac{cm^3}{s}$. HOW FAST IS THE DIAMETER INCREASING WHEN IT IS 50 cm?

SOL:



$$V = \frac{4}{3}\pi r^3 \qquad (r = r(T),\ V = V(T))$$

$$\frac{dV}{dT} = 100 \frac{cm^3}{s}$$

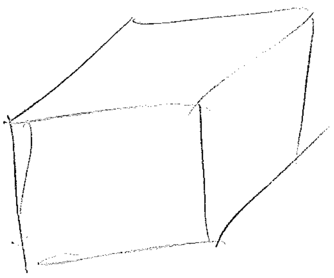
|| ← ALSO EQUAL

$$\frac{d\left(\frac{4}{3}\pi r^3\right)}{dT} = \frac{4}{3}\pi \cdot 3r^2 r' = 4\pi r^2 \cdot r'$$

So $\quad r' = \dfrac{100}{4\pi r^2} = \dfrac{25}{\pi r^2} \qquad D = 2r \quad D' = 2r'$

$$\frac{D'}{2} = \frac{100}{\pi D^2} \underset{\substack{\uparrow \\ D=50}}{=} \frac{2}{\pi(50)^2} \qquad D' = \frac{2}{25\pi}$$

2) AN ICE CUBE IS MELTING. WHEN THE SIDES ARE 3 cm THE RATE OF MELTING IS 2 cm³/min. HOW FAST IS THE LENGTH OF EACH SIDE DECREASING?



ℓ = LENGTH $\qquad (\ell = \ell(T),\ V = V(T))$

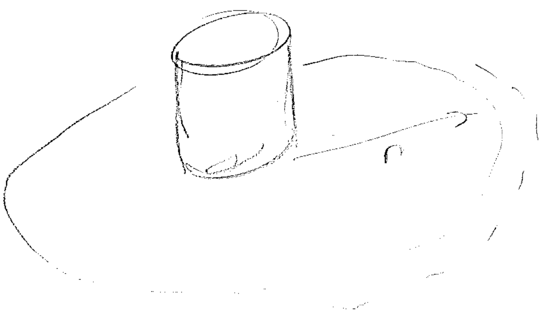
$V = \ell^3$

$$\frac{dV}{dT} = -2 \frac{cm^3}{min} = 3\ell^2 \ell'$$

$$-2 = 3(3)^2 \ell'$$

$$\ell' = -\frac{2}{27} \text{ cm/min}$$

(3) A CANISTER OF OIL IS FORMING A CIRCULAR OIL SLICK AROUND ITSELF. THE AREA INCREASES AT $2\ m^2/min$. HOW FAST IS THE RADIUS OF THE SLICK INCREASING WHEN IT IS 10 m?



$$A = \pi r^2 \qquad (A(T), r(T))$$

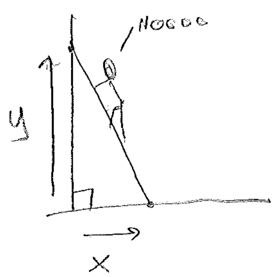
$$\frac{dA}{dT} = 2\pi r r'$$

$$\parallel$$

$$2\ \frac{m^2}{min}$$

$$\pi r r' = 1 \qquad r' = \frac{1}{\pi r} \qquad r = 10 \quad r' = \frac{1}{10\pi}$$

(4) 13 m LADDER HAS ONE END ON GROUND, ONE ON A VERTICAL WALL. THE BOTTOM END SLIPS AWAY AT 3 m/min, HOW FAST IS THE TOP SLIDING DOWN WHEN BOTTOM IS 5 m FROM WALL?



PITHAGORAS: $y^2 + x^2 = 13^2 \qquad (y(T), x(T))$

$$\frac{d}{dT}(y^2 + x^2) = 0 \qquad 2yy' + 2xx' = 0$$

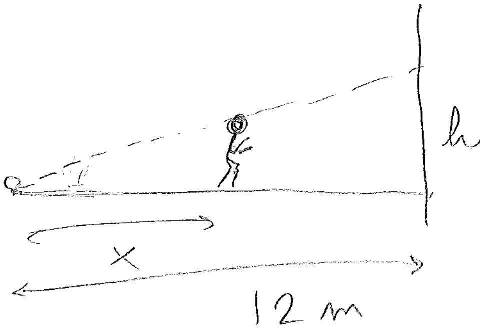
$$yy' = -xx'$$

BUT $x' = 3\ m/min$, $x = 5\ m$

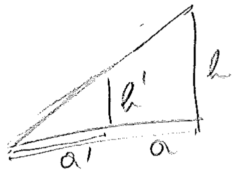
$yy' = -15$ ALSO $y = \sqrt{13^2 - x^2} = \sqrt{169 - 25} = 1$

$$12y' = -15 \qquad y' = \frac{-15}{12} = -\frac{5}{4}\ m/min$$

5) A LAMP SHINES LIGHT ON A WALL 12 m AWAY. A 2 m MAN WALKS TOWARDS THE WALL AT 1.6 m/sec. HOW FAST IS THE HEIGHT OF THE SHADOW DECREASING WHEN HE IS 4 m FROM WALL?



TALES:



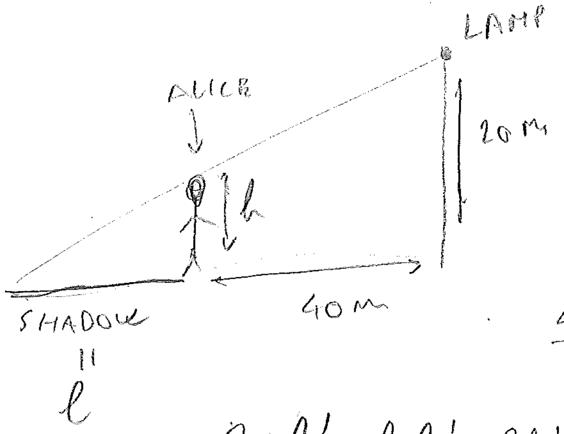
$$\frac{h}{h'} = \frac{a}{a'}$$

$$\left(h(t), x(t)\right)$$

$$\frac{h}{2} = \frac{12}{x} \qquad h = \frac{24}{x} \qquad h' = -\frac{x' \cdot 24}{x^2}$$

$x' = 1.6$ m/s , $x = 8$ m $\underset{(12-4)}{}$ $\qquad h' = \frac{(1.6)(24)}{64} = -\frac{24}{40} = -\frac{6}{10}$

6) ALICE IS IN WONDERAND. A COOKIE MAKES HER GROW AT 0.5 m/min. SHE IS 40 m AWAY FROM A LAMP THAT IS 20 m TALL. HOW FAST IS HER SHADOW INCREASING LENGTH WHEN SHE'S 15 m?



TALES AGAIN: $\left(l(t), h(t)\right)$

$$\frac{20}{h} = \frac{40+l}{l}$$

$$\frac{20}{h} = \frac{40}{l} + 1 \qquad 20l - lh = 40h$$

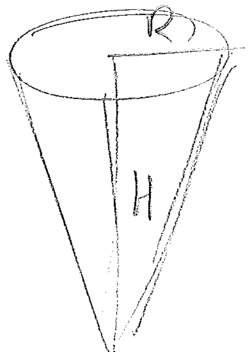
$$20l' - hl' - lh' = 40h' \sim 20l' - 15l' - 60 = 120$$

WE NEED l WHEN $h = 15$

$$\frac{20}{15} = \frac{40}{l} + 1 \sim \frac{1}{3} = \frac{40}{l} \sim l = 120$$

$$\sim 5l' = 80 \sim l' = 16$$

7) A CONE-SHPED ICICLE IS MELTING. ITS VOLUME IS DECREASING BY 10 cm^3/hour, ITS RADIUS BY 0.4 cm/hour. CURRENTLY IT IS 5 cm IN RADIUS AND 14 cm TALL. HOW FAST IS THE HEIGHT CHANGING



$$V = \frac{1}{3}\pi R^2 H \qquad (V(T), R(T), H(T))$$

$$\frac{dV}{dT} = \frac{1}{3}\pi\left(2RR'H + R^2H'\right)$$

$$-10 = \frac{1}{3}\pi\left(2\cdot5\cdot\frac{4}{10}\cdot14 + 5^2H'\right) = \frac{1}{3}\pi\left(56 + 25H'\right)$$

$$-10 + \frac{56\pi}{3} = \frac{25\pi}{3}H' \qquad H' = \frac{-30+56\pi}{25\pi}$$

8) KOOLOG'S MAKS q·1000 PACKS OF FROUT LOOPS CEREAL PER WEEK, THE WHOLESALE PRICE P\$/BOX. THE SUPPLY EQUATION IS

$$6q^2 - 5qp + 2p^3 = 5$$

HOW FAST IS THE SUPPLY OF CEREALS CHANGING WHEN PRICE/BOX IS 6.50 \$, SUPPLY IS 10.000 BOXES AND PRICE/BOX INCREASES BY 0.10 \$/WEE

$$(P(T), q(T)) \qquad 6q^2 - 5qp + 2p^3 = 5$$

$$\frac{d(6q^2 - 5qp - 2p^3)}{dT} = 0 \sim 12qq' - 5(qp' + q'p) - 6p^2p' = 0$$

$$p' = 0.10,\ p = 6.5,\ q = 10 \qquad 120q' - 5(1 + 6.5q') - \frac{6(6.5)^2}{10} =$$

$$(120 - 32.5)q' = 5 + \frac{39(6.5)}{10} \qquad q' = \frac{50 + 39(6.5)}{87.5\cdot10} = -0.23252$$

≈ -233 BOX/WK