WARM-UP QUESTION)

COMPUTE:

① $e^3 f''(3)$, WHERE $f(x) = \dfrac{x+3}{e^x \cdot x}$

② $f'''(3)$, WHERE $f(x) = x^2 \log x$

③ THE VALUE y, WHERE $(0, y)$ IS THE INTERSECTION OF THE TANGENT LINE TO

$$f(x) = \frac{x^2+2}{x+1} \quad \text{AT} \quad a = 2 \quad \text{WITH} \quad x = 0$$

SOL)

① $$f'(x) = \frac{1 \cdot e^x \cdot x - e^x(x+1)(x+3)}{(e^x \cdot x)^2} = \frac{x - (x+1)(x+3)}{e^x x^2} =$$

$$\frac{-x^2 - 3x - 3}{e^x x^2}$$

$$f''(x) = \frac{e^x x^2(-2x-3) - e^x(x^2+2x)(-x^2-3x-3)}{(e^x x^2)^2} = \frac{x^2(-2x-3) + (x^2+2x)(x^2+3x+3)}{e^x x^4}$$

$$= \frac{x^4 + 3x^3 + 6x^2 + 6x}{e^x x^4} = \frac{x^3 + 3x^2 + 6x + 6}{e^x x^3} \quad \text{AT} \quad x = 3$$

$$\frac{68}{27 e^3}$$

So $\dfrac{68}{27}$!

② $f'(x) = 2x \log x + \frac{x^2}{x} = 2x \log x + x$

$f''(x) = 2 \log x + 3 \quad f'''(x) = \frac{2}{x}$ So $\frac{2}{3}$

③ THE LINE IS

$y = f'(2)(x-2) + f(2)$ so

$f'(x) = \frac{2x(x+1) - (x^2+2)}{(x+1)^2} = \frac{x^2+2x-2}{x^2+2x+1} \quad f'(2) = \frac{6}{9}$

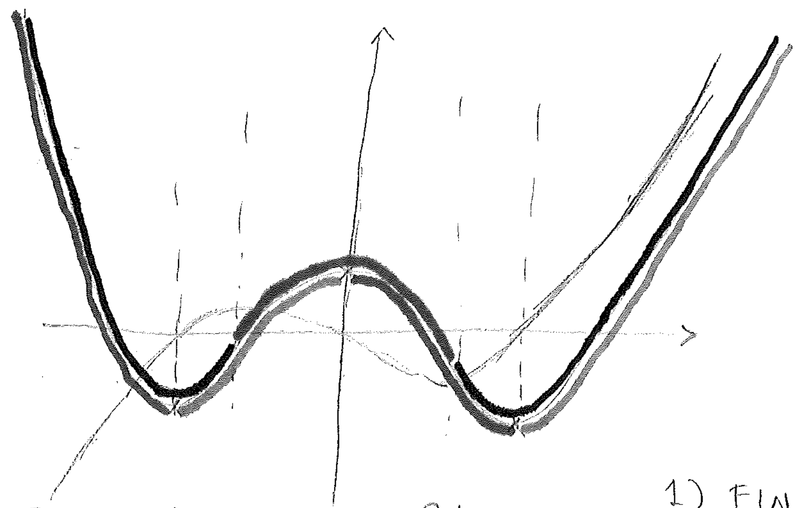
$f(2) = \frac{6}{3} = 2$

So $y = \frac{8}{9}(x-2) + 2$ WHEN $x = 0$

$y = 2 - \frac{16}{9} = \frac{2}{9}$

SKETCHING CURVES (BASIC).

SAY WE HAVE THE GRAPH OF A FUNCTION:

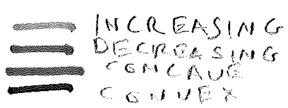


$$f(x) = x^4 - 20x^2 + 64$$

$$f'(x) = 4x^3 - 40x$$

HOW TO SKETCH $f'(x)$

1) FIND INCREASING/DECREASING INTERVAL
2) FIND CONCAVE / CONVEX INTERVALS



THEN

$f'(x)$ IS > 0 WHEN INCREASING
$f'(x)$ IS < 0 WHEN DECREASING

$f'(x)$ IS INCREASING WHEN CONVEX
$f'(x)$ IS DECREASING WHEN CONCAVE

WE CAN GO THE OPPOSITE WAY!

GIVEN $f(x)$ TO SKETCH YOU NEED:

1. ZEROES / SIGN OF $f(x)$
2. " / " " $f'(x)$ (INCREAS/DECR)
3. " / " " $f''(x)$ (CONVEX / CONCAVE)

EXAMPLE: $f(x) = x^3 - 4x$

ZEROES: $0, 2, -2$

$-\quad +\quad -\quad +$

$-2\quad 0\quad 2$

$f'(x) = 2x^2 - 4$

ZEROES $= \pm\sqrt{2} \approx \pm 1{,}4$

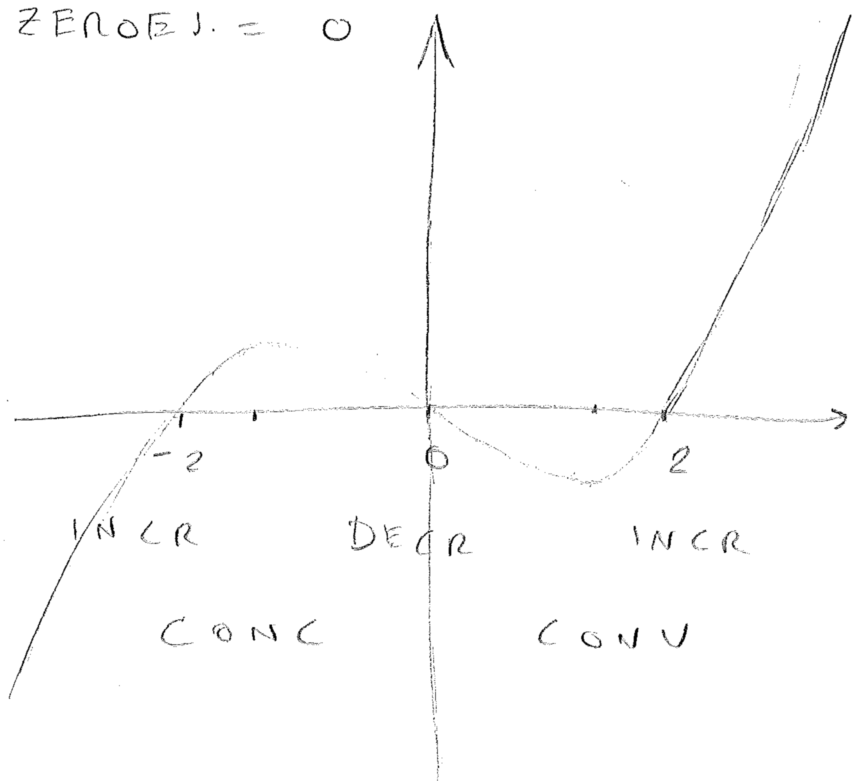
$-\quad +\quad -\quad +$

$-2\quad 0\quad 2$

INCR | DECR | INCR

$f'' = 4x$

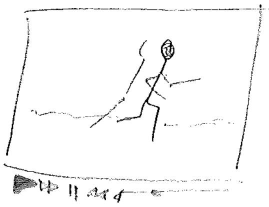
ZEROES. $= 0$



CHAIN RULE

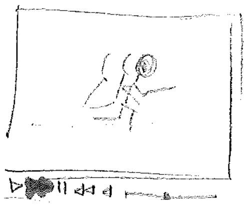
WE NEED ONE LAST RULE TO COMPUTE ALL DERIVATIVES.

Imgine you are looking at a video of a man running with constant acceleration c



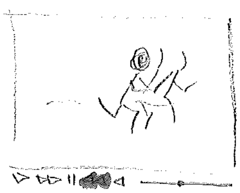
$$v = c\tau$$

YOU PRESS FAST FORWARD $(\tau \to 2\tau)$



$$v = 4c\tau$$

HIS PERCEIVED VELOCITY QUADRUPLES. YOU PRESS FAST REVERSE

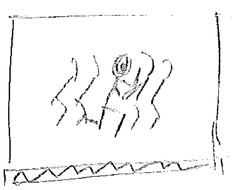


$(\tau \to -2\tau)$

$$v = -4c\tau$$

HIS PERCEIVED VELOCITY QUADRUPLES AND CHANGES SIGN.

FINALLY, WITH AN AD-HOC SOFTWARE YOU MAKE THE VIDEO GO BACK AND FORTH, FOLLOWING A SINE WAVE $(\tau \to \sin\tau)$



$$v = \cos(\tau)\cdot\sin(\tau)\cdot c$$

HIS PERCEIVED VELOCITY IS ... WHAT?

THERE ARE TWO EFFECTS AT PLAY HERE.

1. WE SEE HIM MOVING IN DIFFERENT DIRECTIONS WITH DIFFERENT SPEED
2. WE ARE LOOKING AT A POINT OF THE VIDEO THAT IS $\sin(\tau)$, NOT τ!

"1" "2"

$$\cos(\tau)\cdot\sin(\tau)\cdot c$$

DON'T BELIEVE IT ?

POSITION OF RUNNING MAN $e\frac{t^2}{2}$

COMPOSE WITH $\sin t$ $e\frac{\sin t^2}{2}$

DERIVATIVE $\frac{e(\sin t)'(\sin t)\cdot 2}{2} = e\cos t\sin t$

WE NEED TO BE CAREFUL TAKING THE DERIVATIVE OF A COMPOSITE FUNCTION!

THM: SUPPOSE $g(x)$ IS DIFFERENTIABLE AT (a,b), $f(x)$ IS DIFFERENTIABLE AT $g(x)$ for all x IN (a,b)

THEN:

$f(g(x))$ IS DIFFERENTIABLE AT (a,b)

AND $(f(g(x)))' = g'(x)f'(g(x))$.

NOTE: NOT $g'(x)f'(x)$ (REMEMBER 2), NOT $f'(g'(x))$ (REMEMBER 1)

EXAMPLES:

- $((\sin(x))^n)' = \cos(x)\cdot(n)(\sin x)^{n-1}$

  $g(x) = \sin x$
  $f(x) = x^n$

- IMPORTANT: $(a^x)' = (e^{\ln a\cdot x})' = \ln a\, e^{\ln a\, x} = \ln a\, a^x$

  $g(x) = \ln a\cdot x$
  $f(x) = e^x$

NOTE: IF WE DID NOT KNOW e WE'D DISCOVER IT HERE.

- FACT: $\left(x^{\frac{1}{m}}\right)' = \frac{1}{m} x^{\frac{1}{m}-1} = \frac{1}{m} \sqrt[m]{x^{m-1}}$

$$\left(x^{\frac{n}{m}}\right)' = \frac{n}{m} x^{\frac{1}{m}-1} \cdot x^{\frac{n-1}{m}} = \frac{n}{m} \cdot x^{\frac{n-m}{m}} = \frac{n}{m} x^{\frac{n}{m}-1}$$

$g = x^{\frac{1}{m}}$
$f = x^n$

- FINDING THE DERIVATIVE OF $\log x$

$e^{\log x} = x$ SO $(\log x)' e^{\log x} = (x)' = 1$

SO $(\log x)' x = 1$ THUS $(\log x)' = \frac{1}{x}$

THIS IS A FIRST EXAMPLE OF
IMPLICIT DIFFERENTIATION.

- $f(x) = \ln(3x^6 + x^2)$

$$f'(x) = (3x^6 + x^2)' \frac{1}{3x^6 + x^2} = \frac{18x^5 + 2x}{3x^6 + x^2} = \frac{18x^4 + 2}{3x^5 + x}$$

- $f(x) = x^x \quad (= e^{x \log x})$ ✱ LOGARITHMIC DIFFER.

$$f'(x) = (\log x + 1) e^{x \log x} = (\log x + 1) x^x$$

- $f(x) = \cos(e^x)$ $\quad f'(x) = e^x(-\sin(e^x))$

- $f(x) = \sqrt{1 - x^2}$ $\quad f'(x) = \frac{2x}{2\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

- $f(x) = \sin\left(\frac{360x}{2\pi}\right)$ $\quad f'(x) = \frac{360}{2\pi} \cos(x)$ (Sin in degrees!)

- $$y = \sqrt{x}^{\sqrt{x}} e^{x^2} = e^{\sqrt{x}\log\sqrt{x} + x^2}$$

$$y' = \left(\sqrt{x}\log\sqrt{x} + x^2\right)' e^{\sqrt{x}\log\sqrt{x} + x^2}$$

$$\left(\frac{1}{2\sqrt{x}}\log\sqrt{x} + \cancel{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}\cdot\frac{1}{\cancel{\sqrt{x}}} + 2x\right) e^{\sqrt{x}\log\sqrt{x} + x^2}$$

$$= \left(2x + \frac{\log\sqrt{x} + 1}{2\sqrt{x}}\right)\sqrt{x}^{\sqrt{x}} e^{x^2}$$

CLOSING QUESTION:

CONSIDER

$$f(x) = e^{\sqrt{(\cos x)^2 + x^2}}$$

- $f'(x) = \Big(\underbrace{\sqrt{(\cos x)^2 + x^2}}_{g(x)}\Big)' e^{\sqrt{(\cos x)^2 + x^2}}$
- $g'(x) = \dfrac{2x - 2\cos x \sin x}{2\sqrt{(\cos x)^2 + x^2}}$

SO

$$f'(x) = \frac{2x - 2\cos x \sin x}{2\sqrt{\cos^2 x + x^2}} e^{\sqrt{\cos^2 x + x^2}} = \frac{x - \cos x \sin x}{\sqrt{\cos^2 x + x^2}} e^{\sqrt{\cos^2 x + x^2}}$$

$f'(x) =$

A) $\sqrt{\cos^2 x + x^2}\, e^{\sqrt{\cos^2 x + x^2}}$

B) $\dfrac{x - \cos x \sin x}{\sqrt{\cos^2 x + x^2}} e^{x}$

C) $(x - \cos x \sin x)\, e^{\sqrt{\cos^2 x + x^2}}$

D) $\dfrac{x - \cos x \sin x}{\sqrt{\cos^2 x + x^2}} e^{\sqrt{\cos^2 x + x^2}}$

E) $\dfrac{x - \cos x \sin x}{2\sqrt{\cos^2 x + x^2}} e^{\sqrt{\cos^2 x + x^2}}$

?

GENERALIZED DIFFERENTIATION RULES

WRITE $u = g(x)$

1. $$\frac{d(u^m)}{dx} = m \cdot u^{m-1} \cdot u'$$

2. $$\frac{d}{dx} e^u = u' \cdot e^u$$

3. $$\frac{d}{dx}(\sin u) = u' \cdot \cos u$$

4. $$\frac{d(\cos u)}{dx} = -u' \cdot \sin u$$

5. $$\frac{d}{dx}(\tan u) = \frac{u'}{\cos^2 u}$$

6. $$\frac{d(\log u)}{dx} = \frac{u'}{u}$$