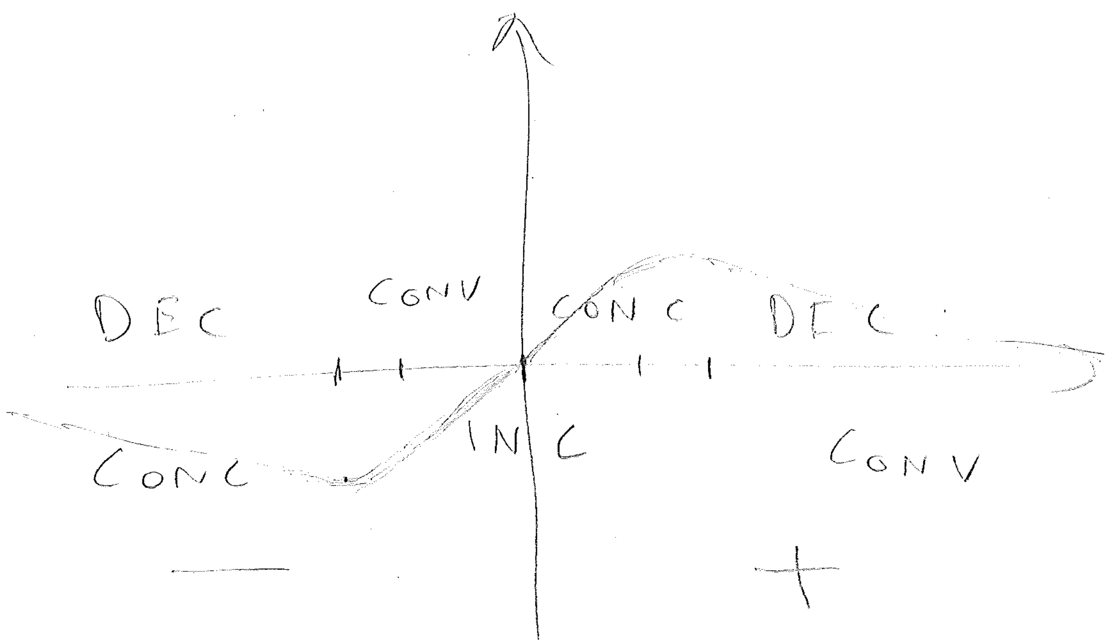
WARM-UP

$f(x) = x e^{-x^2}$ SKETCH $f(x)$

$f'(x) = e^{-x^2}(-2x^2+1)$ $\pm\sqrt{\frac{1}{2}}$

$f''(x) = e^{-x^2}(4x^3-6x)$ $0, \pm\sqrt{\frac{3}{2}}$



GENERALIZED DIFFERENTIATION RULES

WRITE $u = g(x)$

1) $\frac{d(u^m)}{dx} = m \cdot u^{m-1} \cdot u'$

2) $\frac{d}{dx} e^u = u' \cdot e^u$

3) $\frac{d}{dx}(\sin u) = u' \cdot \cos u$

4) $\frac{d(\cos u)}{dx} = -u' \cdot \sin u$

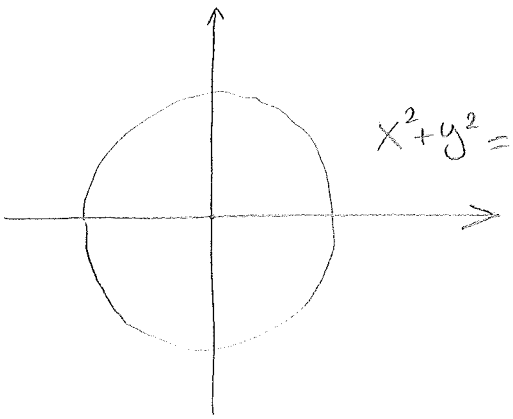
5) $\frac{d}{dx}(\tan u) = \frac{u'}{\cos^2 u}$

6) $\frac{d(\log u)}{dx} = \frac{u'}{u}$

IMPLICIT DIFFERENTIATION

QUESTION:

HOW DO WE FIND THE TANGENT TO A POINT IN A CIRCLE?



$x^2+y^2=1$

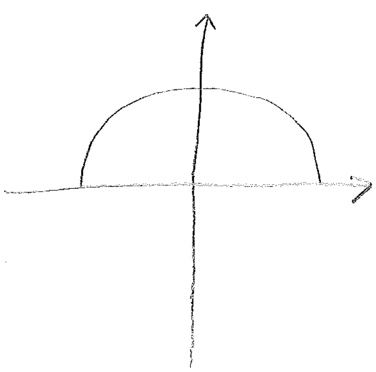
WE CAN PARAMETRIZE THESE POINTS

$$y=\sqrt{1-x^2}$$

SO WE HAVE $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

AND WE CAN FIND THE TANGENT.

<u>BUT</u> WE MADE A CHOICE HERE! WE CHOSE TO PARAMETRIZE ONLY <u>HALF</u> OF THE CIRCLE, SO FOR NEGATIVE y WE NEED $y=-\sqrt{1-x^2}$



AND MOREOVER THIS DOES NOT REALLY WORK FOR $x=\pm 1$

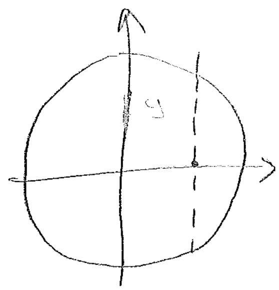
HAS MATH FAILED US?

NOT REALLY! IT'S US TRYING TO USE FUNCTIONS IN THE WRONG WAY

BY DEFINITION A FUNCTION TAKES A VALUE x IN THE DOMAIN AND GIVES US

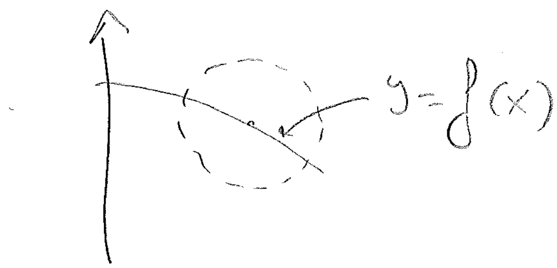
A <u>UNIQUE</u> VALUE IN THE CODOMAIN!

SOME EQUATIONS, SUCH AS THE EQUATION OF A CIRCLE, DO NOT ADMIT A <u>UNIQUE</u> SOLUTION

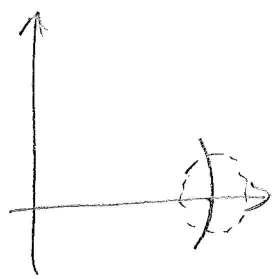


$y^2 = \frac{1}{2}$ $\quad x = \pm\frac{\sqrt{2}}{2}$

<u>BUT</u> THEY DO IF WE ONLY <u>LOOK CLOSE</u> TO A CERTAIN SOLUTION



EVEN WORSE, IT MAY BE THAT WE CAN NEVER FIND <u>ALL</u> SOLUTIONS CLOSE TO A SPECIFIC ONE AS A <u>FUNCTION OF X</u> !



SO ARE WE <u>FORCED</u> TO CHOOSE <u>A PRIORI</u> A SUBSET OF THE SOLUTIONS THAT CAN BE WRITTEN $y = f(x)$ TO GET <u>INFORMATION</u> ?

NO! SUPPOSE WE KNOW THAT <u>CLOSE TO A SOLUTION</u> (x_0, y_0) WE THRE IS f S.T. $y = f(x)$ DESCRIBES THE SOLUTIONS TO OUR EQUATION. THEN WE CAN WRITE $y' = \frac{dy}{dx}$ AND OBTAIN THE DERIVATIVES, TANGENTS ETC.

EXAMPLE)

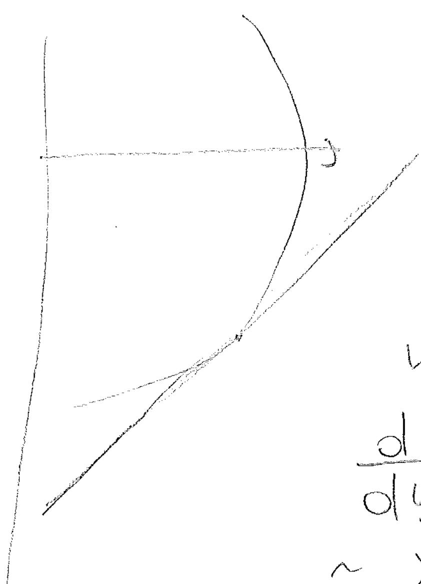
WE WANT TO FIND THE TANGENT TO THE CIRCLE $x^2+y^2=1$ AT A POINT $(x_0,y_0)\neq(\pm 1,0)$

WE CAN <u>DIFFERENTIATE THE EQUATION</u>

$$\frac{d(x^2+y^2)}{dx}=\frac{d\,1}{dx} \sim 2x+2y'y=0$$

$$\sim 2y'y=-2x \sim y'=\frac{-x}{y}$$ SO THE SLOPE AT (x_0,y_0) IS $y'(x_0)=\frac{-x_0}{y_0}$!

FOR EXAMPLE $(x_0,y_0)=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ THEN THE TANGENT IS $y=x-\sqrt{2}$!



THIS WORKS BOTH FOR NEGATIVE AND POSITIVE y! WHAT ABOUT $(\pm 1,0)$? WE WILL HAVE TO WRITE $x=f(y)$ SO

$$\frac{d}{dy}(x^2+y^2)=0 \sim 2xx'+2y=0$$

$$\sim x'=\frac{-y}{x}$$ AT $(1,0)$ WE GET $x'=0$ SO THE TANGENT IS

$$x=y\cdot 0+1 \sim x=1 \;!$$

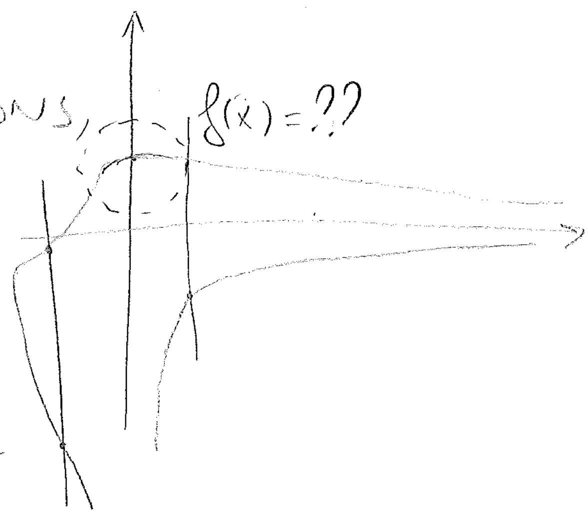
THE MOST IMPORTANT FACT IS THAT THIS TECHNIQUE <u>WORKS</u> EVEN WHEN WE <u>CANNOT</u> WRITE $y=f(x)$ <u>EPLICITLY!</u> WE JUST NEED TO KNOW THAT AN f SOLVING OUR EQUATION <u>EXISTS</u> CLOSE TO SOME SOLUTION (x_0, y_0).

<u>EXAMPLE:</u>

WE ARE LOOKING AT THE EQUATION

$$y^3 + x^2 y^4 = 1 + 2x$$

THE SET OF SOLUTIONS, PICTURED HERE, <u>FAILS AT BEING A FUNCTION</u> AND IT DOES SO <u>BADLY</u>.



MOREOVER, WE CAN'T WRITE DOWN AN EPLICIT SOLUTION $y = f(x)$.

BUT! WE KNOW THAT $(0,1)$ IS A SOLUTION. CAN WE FIND $f'(0)$?

$$\frac{d}{dx}(y^3 + x^2y^4) = \frac{d}{dx}(1+2x) \sim 3y'y^2 + 2xy^4 + 4x^2y^3y' = 2$$

$$y'(3y^2 + 4x^2y^3) = 2 - 2xy^4 \sim y' = \frac{2-2xy^4}{3y^2+4x^2y^3}$$

SO AT $(0,1)$ $y' = \frac{2}{3}$!

EXAMPLE: LOGARITHMIC DIFFERENTIATION

SOMETIMES THE "IMPLICIT APPROACH" CAN HELP US ALSO IN THE EXPLICIT CASE.

SAY WE HAVE TO FIND THE DERIVATIVE OF $y = x^x$. THEN WE CAN EXTRACT THE LOGARITHM

$\ln(y) = \ln(x^x) = x\ln(x)$, DIFFERENTIATE,

$$\frac{d\ln(y)}{dx} = \frac{d}{dx} x\ln(x) \sim \frac{y'}{y} = 1 + \ln(x)$$

SOLVE FOR y'

$y' = y(1+\ln(x))$ AND SUBSTITUTE $y = x^x$

$y' = x^x(1+\ln(x))$

IN GENERAL

$y = u(x)^{v(x)} \sim \ln y = v(x)\ln(u(x))$ DIFFERENTIATE

$$\frac{y'}{y} = \frac{v(x)}{u(x)} + v'(x)\ln(u(x)) \sim y' = y\left(\frac{v(x)}{u(x)} + v'(x)\ln(u(x))\right)$$

SUBSTITUTE $y' = u(x)^{v(x)}\left(\frac{v(x)}{u(x)} + v'(x)\ln(u(x))\right)$

ANOTHER EXAMPLE

$$y = \frac{u(x)^n v(x)^m}{h(x)^k} \sim \ln(y) = n\ln(u(x)) + m\ln(v(x)) - k\ln(h(x))$$

$$\frac{y'}{y} = n\frac{u'(x)}{u(x)} + m\frac{v'(x)}{v(x)} - k\frac{h'(x)}{h(x)} \quad y' = y\left(n\frac{u'(x)}{u(x)} + m\frac{v'(x)}{v(x)} - k\frac{h'(x)}{h(x)}\right)$$

So $y' = \frac{u(x)^n v(x)^m}{h(x)^m}\left(n\frac{u'(x)}{u(x)} + m\frac{v'(x)}{v(x)} - k\frac{h'(x)}{h(x)}\right)$

EXAMPLES:

- $f(x) = \sqrt{x}^{(\cos x)^2}$ $\quad f'(x) = \sqrt{x}^{(\cos x)^2}\left(\frac{\cos(x)^2}{\sqrt{x}} - 2\cos x \sin x \underbrace{\ln(\sqrt{x})}_{\frac{1}{2}\ln(x)}\right)$

- $f(x) = \frac{\sin(x)^3 \ln(x)^3}{\cos(x)^2}$ $\quad f'(x) = \frac{\sin(x)^3 \ln(x)^3}{\cos(x)^2}\left(3\frac{\cos x}{\sin x} + 3\frac{1}{x\ln x} + 2\frac{\sin x}{\cos x}\right)$

HIGHER ORDER IMPLICIT DERIVATIVES

WITH SOME EFFORT WE CAN COMPUTE HIGHER IMPLICIT DERIVATIVES.

EXAMPLE:

- GIVEN $x^2 + y^2 = 1$ FIND A FORMULA FOR y''

WE KNOW SINCE EARLIER THAT $y' = \frac{-x}{y}$. THEN

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{-y + y'x}{y^2} \overset{\text{SUBSTITUTE } y'}{=} \frac{-y - \frac{x^2}{y}}{y^2}$$

ONE MORE EXAMPLE: $y^{2/3} + x^{\frac{2}{3}} = 8$. FIND y' AT $(8,8)$.

$y'\frac{2}{3}y^{\frac{2}{3}-1} + \frac{2}{3}x^{\frac{2}{3}-1} = 0 \sim y' = -\frac{x^{\frac{2}{3}-1}}{y^{\frac{2}{3}-1}}$

So AT $(8,8)$ $y' = -\frac{8^{-\frac{1}{3}}}{8^{-\frac{1}{3}}} = -1$.