INSTANTANEOUS RATE OF CHANGE:

THE DERIVATIVE

WE HAVE SEEN ALREADY HOW TO COMPUTE THE INSTANTANEOUS CHANGE / SLOPE OF THE TANGENT LINE OF A FUNCTION $f(x)$.
WE HAVE ALSO DISCOVERED THAT A NECESSARY CONDITION FOR $f(x)$ TO ADMIT AN INSTANTANEOUS RATE OF CHANGE AT $\bar{x}$ IS THAT $f(x)$ MUST BE CONTINUOUS AROUND $\bar{x}$, THAT IS, CONTINUOUS ON AN OPEN INTERVAL (a, b) CONTAINING $\bar{x}$. WE ARE READY TO GIVE A FORMAL DEFINITION:

DEFINITION:

LET $f(x)$ BE A CONTINUOUS FUNCTION ON AN OPEN INTERVAL (a, b). LET $\bar{x}$ BE A POINT IN (a, b). IF THE LIMIT

$$\lim_{h \to 0} \frac{f(\bar{x}+h) - f(\bar{x})}{h}$$

EXISTS, WE SAY THAT $f(x)$ IS DIFFERENTIABLE AT $\bar{x}$, AND WE CALL THE LIMIT THE DERIVATIVE OF $f(x)$ AT $\bar{x}$, WRITTEN

$$* \quad \frac{df}{dx}(\bar{x}) \quad \text{OR} \quad f'(\bar{x}) \quad \text{OR} \quad \dot{f}(\bar{x})$$

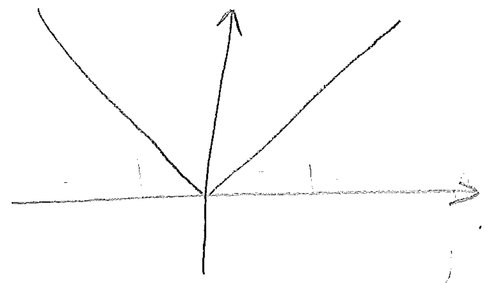
IF $f(x)$ IS DIFFERENTIABLE EVERYWHERE IN (a, b) WE SAY $f(x)$ IS DIFFERENTIABLE ON (a, b).

* if $f(x) = y$ $\frac{df}{dx}$ can also be written $\frac{dy}{dx}$

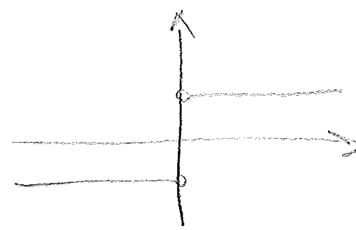
NOT ALL CONTINUOUS FUNCTION ARE DIFFERENTIABLE!

EXAMPLES:

- $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$



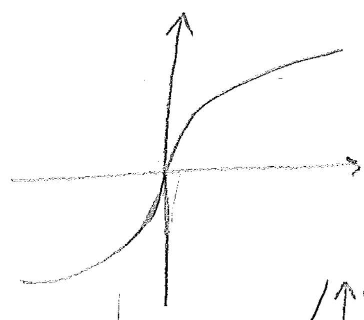
$f'(x) = \begin{cases} +1 & x > 0 \\ \text{NOT DEFINED} & x = 0 \\ -1 & x < 0 \end{cases}$



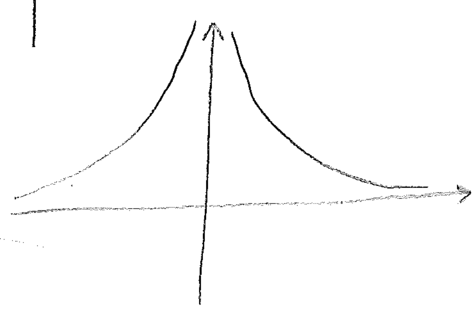
WHY EXACTLY?

$$\frac{|x+h| - |x|}{h} = \begin{cases} \frac{h}{h} & x > 0 \\ \frac{-h}{h} & x < 0 \\ \frac{|h|}{h} & x = 0 \end{cases}$$

- $f(x) = \sqrt[3]{x}$

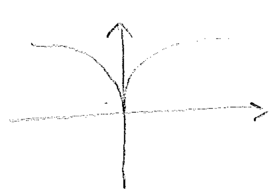


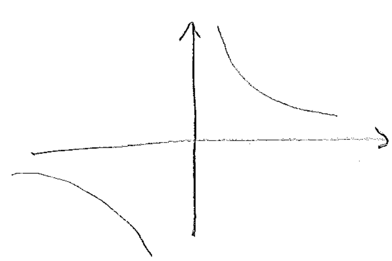
$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$



WHY EXACTLY? WE'LL SEE!

- $f(x) = \sqrt[3]{|x|}$
- $f'(x) = \begin{cases} \frac{1}{3\sqrt[3]{x^2}} & x > 0 \\ -\frac{1}{3\sqrt[3]{x^2}} & x < 0 \end{cases}$





IF f IS DIFFERENTIABLE ON SOME INTERVAL (a, b) WE CAN SEE ITS DERIVATIVE AS A FUNCTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

AS AN EXAMPLE, WE WILL COMPUTE THE DERIVATIVE OF $f(x) = x$ AND $f(x) = x^2$ AS A FUNCTION OF x

- $f(x) = x$ $\quad f'(x) = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

- $f(x) = x^2$ $\quad f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 - x^2 + 2hx + h^2}{h}$

$$= \lim_{h \to 0} \frac{2hx + h^2}{h} = 2x + \lim_{h \to 0} h = 2x.$$

HOW ABOUT CONSTANTS?

- $f(x) = a$ $\quad f'(x) = \lim_{h \to 0} \frac{a - a}{h} = 0$

LINEAR FUNCTIONS?

- $f(x) = ax + b$ $\quad f'(x) = \lim_{h \to 0} \frac{a(x+h) + b - (ax + b)}{h}$

$$= \lim_{h \to 0} \frac{a(x+h) - ax}{h} + \lim_{h \to 0} \frac{b - b}{h} = a$$

$$\| $$

$$a\left(\frac{d(x)}{dx}\right) + \frac{d(b)}{dx}$$

WHAT WE CAN DEDUCE FROM THE REASONING ABOVE:

- CONSTANTS MULTIPLYING A FUNCTION GET OUT OF THE DERIVATIVE JUST LIKE THEY DO FOR LIMITS, THAT IS

  - $$\frac{d\, a\cdot f}{dx} = a\frac{df}{dx}$$ OR EQUIVALENTLY

  - $$(af)' = af'$$

- THE DERIVATIVE OF A SUM OF FUNCTIONS IS THE SUM OF THE RESPECTIVE DERIVATIVES, THAT IS

  - $$\frac{d(f+g)(x)}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$$ EQUIV:

  - $$(f+g)' = f' + g'$$

THESE TWO RULES GREATLY SIMPLIFY OUR LIVES, JUST LIKE FOR LIMITS! LET'S TRY TO FIND MORE: LET'S TRY TO COMPUTE THE DERIVATIVE OF

$f(x) = x^3$

- $$f'(x) = \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0}\frac{x^3 - x^3 + 3hx^2 + 3h^2x + h^3}{h}$$

$$= 3x^2 + \lim_{h\to 0} 3hx = 3x^2$$

THIS LOOKS LIKE IT SHOULD HAPPEN FOR ALL m...

- LET $f(x)$ BE x^m. THEN

— $\frac{df(x)}{dx} = \frac{dx^m}{dx} = mx^{m-1}$ EQUIV:

— $f'(x) = (x^m)' = mx^{m-1}$

WE WILL SEE THAT THIS RULE IS IMPLIED BY A STRONGER ONE.

BEFORE VENTURING FURTHER, LET'S SEE A LIST OF DERIVATIVES WE KNOW

FUNCTION	DERIVATIVE
$f(x)$	$f'(x)$
c	0
x^m	mx^{m-1}
e^x	e^x (!)
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

WAIT: $\frac{de^x}{dx} = e^x$? SO THE EXPONENTIAL FUNCTION IS EQUAL TO ITS OWN RATE OF CHANGE! ARE THERE OTHERS?

THEOREM: IF $f'(x) = f(x)$, $f(x) = \lambda e^x$, λ IN $\mathbb{R}$.

THIS GIVES US AN ALTERNATIVE DESCRIPTION OF THE FUNCTION e^x (AND THE NUMBER e)

e IS THE ONLY NUMBER a SUCH THAT

$$* \; f(x) = a^x \text{ SATISFIES } f(x) = f'(x).$$

* How is a^x defined? On rational numbers $a^{\frac{p}{q}} = \sqrt[q]{a^p}$, Then $a^x = \lim_{\frac{p}{q} \to x} a^{\frac{p}{q}}$.

WHY DO WE CARE ABOUT THIS?

$f(x) = f'(x)$ IS THE FIRST EXAMPLE OF A DIFFERENTIAL EQUATION.

D.E. (AND EXPONENTIAL FUNCTIONS) APPEAR WHENEVER WE HAVE A FUNCTION WHOSE RATE OF CHANGE (AND RATE OF CHANGE OF RATE OF CHANGE, AND SO ON...) ARE TIED TO THE FUNCTION BY AN EQUATION

EXAMPLES:
DRAG, POPULATION GROWTH, BATTERY DISCHARGE, ECONOMICS...

BACK TO DERIVATIVE RULES! THE FOLLOWING TWO RULES ARE NOT AS INTUITIVE AS THE PREVIOUS ONE.

PRODUCT RULE

LET $f(x)$, $g(x)$ BE DIFFERENTIABLE. THEN

- $$\frac{d\, f(x)g(x)}{dx} = f(x)\frac{d\,g(x)}{dx} + g(x)\frac{d\,f(x)}{dx}$$
- $$\left(f(x)g(x)\right)' = f(x)g'(x) + g(x)f'(x)$$

WE ARE NOT GOING TO SHOW WHY THIS WORKS; NOTE THAT IT IMPLIES THE RULE FOR POWER FUNCTIONS ONCE ONE KNOWS THE DERIVATIVE OF $f(x) = x$

$$(x^n)' \underset{\text{PROD RULE}}{=} x(x^{n-1})' + x^{n-1} = x \cdot \left((n-2)x^{n-2}\right) + x^{n-1}$$

$$= x^{n-1} \cdot (n-2+1) = (n-1)x^{n-1}$$

WE CAN ALREADY COMPUTE THE DERIVATIVE OF A LOT OF FUNCTIONS!

$f(x) = e^x(x^3+3x+5)$ $\quad f'(x) = (e^x)'(x^3+3x+5) + e^x(x^3+3x+5)' = e^x(x^3+3x+5) + e^x(3x^2+3) = e^x(x^3+3x^2+3x+8)$

A NOTE OF CAUTION:

THE PRODOUCT RULE ONLY APPLIES WHEN f, g ARE DIFFERENTIABLE! IT DOES NOT TELL YOU, FOR INSTANCE, THAT IF f IS NOT DIFF AT I AND g IS, THEN fg IS NOT;

- $f(x) = |x| \quad g(x) = x \quad fg(x) = |x|x = \begin{cases} -x^2 & x<0 \\ x^2 & x \geq 0 \end{cases}$

$$\lim_{h\to 0^-} \frac{|h|h}{h} = \lim_{h \to 0^+} \frac{|h|h}{h} = 0$$

SO $(fg)' = \begin{cases} -2x & x<0 \\ 2x & x \geq 0 \end{cases}$ IS PERFECTLY WELL DEFINED AT 0!

I.V.T. EXERCISE 1

Suppose: $f(x)$ is defined on $[-1, 1]$

$f(x)$ is continuous ON $(-1, 0), [0, 1]$

$f(-1) = -3$ $\lim_{x \to 0^-} f(x) = 1$ $f(\frac{1}{2}) = \frac{1}{4}$

$\lim_{x \to -1^+} f(x) = 2$ $f(0) = 0$ $f(1) = -1$

Q 1: THE IVT TELLS US

A) There is an x S.T. $f(x) = -2.5$ IN $[-1, 1]$
B) There is an x S.T. $f(x) = -0.7$ IN $[0, 1]$
C) There is an x S.T. $f(x) = \frac{1}{2}$ IN $[-\frac{1}{2}, \frac{1}{2}]$
D) Nothing, we can't apply I.V.T.
E) None of The above

Q 2: THE IVT <u>DOES NOT</u> TELL US

A) There is an x S.T. $f(x) = \frac{3}{2}$ IN $(-1, 0)$
B) There is an x S.T. $f(x) = \frac{1}{8}$ IN $(0, \frac{1}{2})$
C) There is an x S.T. $f(x) = 0$ IN $(\frac{1}{2}, 1]$
D) There is an x S.T. $f(x) = \frac{3}{2}$ IN $[-0.99, -0.01]$
E) There are at least Two points where $f(x)$ is 0

I.V.T EXERCISE 2

CONSIDER THE FUNCTION, DEPENDING ON CONSTANTS

c_1, c_2

$$f(x) = \begin{cases} c_1 x^2 + c_2 & x < 1 \\ e^x & x \geq 1 \end{cases}$$

FIND c_1, c_2 S.T. THE I.V.T., APPLIED TO

f IN THE INTERVAL $[-2, 2]$ TELLS US

THAT THERE IS x S.T. $f(x) = 10$

1) $f(x)$ MUST BE CONTINUOUS

$$c_1 + c_2 = e$$

2) $f(-2)$ MUST BE GREATER THAN 10 (AS $e^2 < 10$)

$$4c_1 + c_2 \geq 10$$

So $c_1 = -c_2 + e$

$$4(-c_2 + e) + c_2 \geq 10 \sim -3c_2 \geq 3 - 4e$$

$$c_2 \leq \frac{10 - 4e}{-3}$$ positive

example : $c_1 = e$, $c_2 = 0$

Then f continuous, $f(-2) = 4e > 10$, $f(2) = e^2 < 10$

Q3: WHICH OF THESE SOLVES THE PROBLEM ABOVE?

A) $c_1 = \frac{e}{2}$, $c_2 = \frac{e}{2}$

B) $c_1 = 0$, $c_2 = e$

C) $c_1 = e$, $c_2 = -e$

D) $c_1 = e + \frac{1}{100}$, $c_2 = 0.01$

E) $c_1 = e - \frac{1}{100}$, $c_2 = 0.01$

GLUING DERIVATIVES 1

CONSIDER THE FUNCTION:

$$f(x) = \begin{cases} c_1 x & x < 0 \\ e^x - c_2 & x \geq 0. \end{cases}$$

FIND c_1, c_2 S.T. $f(x)$ IS DIFFERENTIABLE EVERYWHERE

Q4: WHICH OF THESE IS A SOL:

A) 0, 1
B) 1, 0
C) 0, 0
D) 1, 1
E) 2, 2

SOL:

f CONTINUOUS IMPLIES $e^0 - c_2 = 0$ SO $c_2 = 1$

f DIFF AT 0 IMPLIES $(e^x - 1)' = (c_1 x)'$ AT 0.

SO $e^0 = c_1$